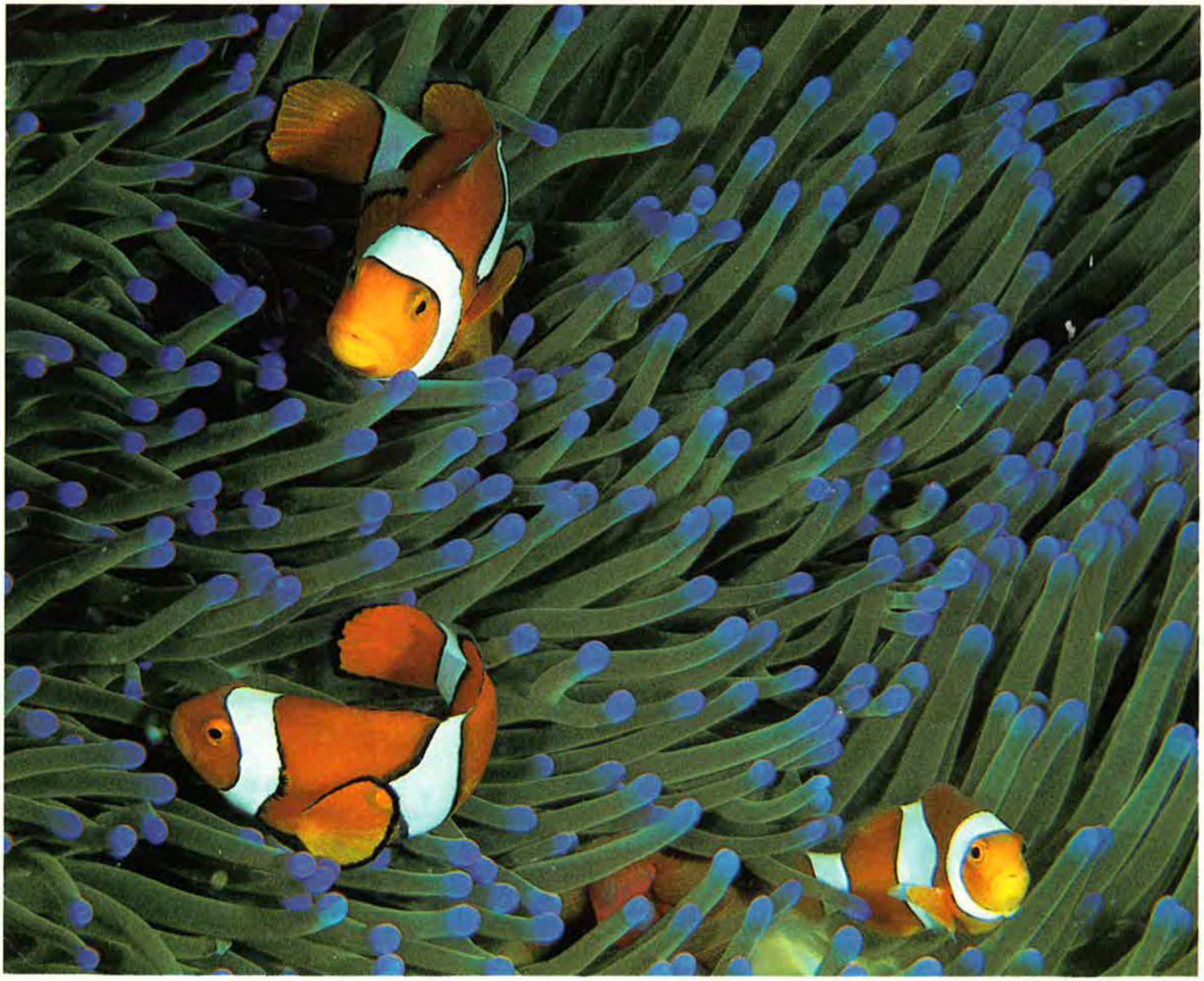
ADVANCED ALGEBRA ADVANCED MATHEMATICS

TEACHER'S EDITION

# Modeling with Logarithms

JACK BURRILL, MIRIAM CLIFFORD, JAMES LANDWEHR

DATA-DRIVEN MATHEMATICS



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Jack Burrill, Miriam Clifford, and James M. Landwehr

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# Table of Contents

# About *Data-Driven Mathematics*

Historically, the purposes of secondary-school mathematics have been to provide students with opportunities to acquire the mathematical knowledge needed for daily life and effective citizenship, to prepare students for the workforce, and to prepare students for postsecondary education. In order to accomplish these purposes today, students must be able to analyze, interpret, and communicate information from data.

*Data-Driven Mathematics* is a series of modules meant to complement a mathematics curriculum in the process of reform. The modules offer materials that integrate data analysis with high-school mathematics courses. Using these materials helps teachers motivate, develop, and reinforce concepts taught in current texts. The materials incorporate the major concepts from data analysis to provide realistic situations for the development of mathematical knowledge and realistic opportunities for practice. The extensive use of real data provides opportunities for students to engage in meaningful mathematics. The use of real-world examples increases student motivation and provides opportunities to apply the mathematics taught in secondary school.

The project, funded by the National Science Foundation, included writing and field testing the modules, and holding conferences for teachers to introduce them to the materials and to seek their input on the form and direction of the modules. The modules are the result of a collaboration between statisticians and teachers who have agreed on the statistical concepts most important for students to know and the relationship of these concepts to the secondary mathematics curriculum.

A diagram of the modules and possible relationships to the curriculum is on the back cover of each Teacher's Edition of the modules.

# Using This Module

## Why the Content Is Important

There are many patterns in the world that can be described by mathematics. Mathematical modeling is the process of finding, describing, analyzing, and evaluating such patterns using mathematics. The first step in building such a model is to recognize different categories of patterns and to understand the underlying mathematical structure within those categories that can help in the search for an appropriate mathematical model.

In this module, students explore ways to find a mathematical model for problems involving bivariate data. They use data sets such as the federal debt over time, decibel measures from various sounds, and the number of motor vehicles registered in the United States to investigate similarities and differences among patterns. They study the effects of scale changes and transformations on data plots and on the graphs of various mathematical functions. Logarithms are introduced graphically and numerically in a non-traditional way that emphasizes their role in mathematical modeling. Students use their algebra skills and concepts developed in the module to create mathematical models. These models are used to answer questions, summarize results, and make predictions about variables. Correlation is introduced as an assessment of the linear relationship between two variables and as an aid in the modeling process. This module should be taught in an advanced algebra or precalculus course in conjunction with a unit on exponential and logarithmic functions.

*Modeling with Logarithms* is divided into three units.

### Unit I: Patterns and Scale Changes

Mathematicians and statisticians represent and examine data patterns in different forms: numeric, geometric (graphs or pictures), and symbolic (formulas). Each representation yields different information and aids in understanding. The interpretation of a data pattern may also be affected by the scale or units. Changing the units from centimeters to meters in a data set changes the appearance of the number pattern or graph, which can influence the message a data set conveys. Lesson 1 is devoted to the study of patterns in data and their representations. Lesson 2 examines the effects of unit or scale change upon the graphic representation of the data.

### Unit II: Functions and Transformations

There are some fundamental functions one should be familiar with in both symbolic and graphic form. Often the graph of a function can be altered in a way that would make the process of mathematical modeling simpler. As students relate the shape of the graph with the equation of a function, they will learn to use functions to transform data which alters the graphic representation. Lesson 3 reviews the relationships among some very useful functions, their symbolic expressions, and their graphs. Lesson 4 examines patterns in graphs and deals with the concepts of increasing, decreasing, linear, and nonlinear functions. Lesson 5 is concerned with the transformation of data to linearize a scatter plot. Lesson 6 investigates the changes in a graph relative to inverse functions.

### Unit III: Mathematical Models from Data

Several tools can be used to create a mathematical model and analyze how well a model describes a data set. These include the ideas already studied: looking for patterns in functions, transforming data, and considering scale changes. The mathematical model is the most appropriate equation that fits a data set. In searching for a model, you may find more than one that seem appropriate. It is therefore necessary to develop some skills to help determine which is the best one. Lessons 7–10 address the concepts involved in determining "best fit." In Lesson 11, all the modeling skills must be used in an application.

### Content

Mathematics content: Students will be able to:

- Find mathematical patterns from data and graphs.
- Linear and nonlinear changes of scale.
- Work with power, polynomial, logarithmic, exponential, and inverse functions.
- Create mathematical models for bivariate data.
- Solve equations.

Statistics content: Students will be able to:

- Construct scatter plots.
- Create linear and nonlinear models for bivariate data.
- Use least-squares regression and residuals.
- Use the concept of correlation.

## Instructional Model

The instructional emphasis in *Modeling with Logarithms,* as in all of the modules in *Data-Driven Mathematics,* is on discourse and student involvement. Each lesson is designed around a problem or mathematical situation and begins with a series of introductory questions or scenarios that can be used to prompt discussion and raise issues about that problem. These questions can provoke students' involvement in thinking about the problem and help them understand why such a problem might be of interest to someone in the world outside of the classroom. The questions can be used as part of whole-class discussions or by students working in groups. In some cases, the questions are appropriate to assign as homework to be done with input from families or from others not part of the school environment.

These opening questions are followed by the presentation of some discussion issues that clarify the initial questions and begin to shape the direction of the lesson that follows. Once the stage has been set for the problem, students begin to investigate the situation mathematically. As students work their way through the investigations, it is important that they have the opportunity to share their thinking with others and to discuss their solutions in small groups and with the whole class. Many of the exercises are designed for groups, where each member of the group does one part of the problem and the results are compiled for a final analysis and solution. Multiple solutions and solution strategies are also possible; it is important for students to recognize these situations and to discuss the reasoning that leads to different approaches. This will provide each student with a wide variety of approaches from which to build their own understanding of the mathematics.

In many cases, students are expected to construct their own understanding after being asked to think about the problem from several perspectives. They do need, however, validation of their thinking and confirmation that they are on the right track, which is why discourse among students and between students and teacher is critical. In addition, an important part of the teacher's role is to help students tie the ideas within an investigation together and to provide an overview of the "big picture" of the mathematics within the investigation. To facilitate this, a review and formalization of the mathematics is presented in a summary box following each investigation.

Each investigation is followed by a Practice and Applications section where students can revisit ideas presented within the lesson. These exercises may be assigned as homework, given as group work during class, or omitted altogether if students seem to be ready to move on.

At the end of each unit, assessment lessons are included within the student text. These lessons can be assigned as long-range take-home tasks, as group assessment activities, or as regular classwork. The ideas within the assessment provide a summary of the unit activities and can serve as a valuable way to enable students to demonstrate what they know and can do with the mathematics. It is helpful to pay attention to the strategies students use to solve a problem. This knowledge can be used as a way to help students grow in their ability to apply different strategies and learn to recognize those strategies that will enable them to find solutions efficiently.

## Prerequisites

Students should be able to graph bivariate data, identify patterns in scatter plots, plot functions, manipulate algebraic expressions involving two variables, use exponents, fit a straight line to bivariate data, and solve equations. A thorough understanding of logarithms is not essential, but students should be familiar with the basic properties of logarithmic and exponential functions.

## Pacing/Planning Guide

There is a logical progression through the lessons. Depending on the background of the individual students, some lessons could be shortened.

| LESSON | OBJECTIVE | PACING |
|---|---|---|
| **Unit I: Patterns and Scale Changes** | | |
| Lesson 1: Patterns | Understand how ordered pairs, graphs, equaations, and tables can be used to describe patterns. | 2 class periods |
| Lesson 2: Changes in Units on the Axes | Understand how changes in units affect tables and graphs. | 2 class periods |
| Assessment: Speed Versus Stopping Distance and Height Versus Weight | | varies |
| **Unit II: Functions and Transformations** | | |
| Lesson 3: Functions | Recognize graphs and equations for different functions. | 2 class periods |
| Lesson 4: Patterns in Graphs | Define a mathematical model and explore different data sets; identify which data sets can be represented by linear and nonlinear models; make suggestions regarding probable models. | 1 class period |
| Lesson 5: Transforming Data | Transform specific data sets to make them appear linear when plotted in a scatter plot. | 3 class periods |
| Lesson 6: Exploring Changes on Graphs | Recognize and understand how the shape of a graph changes when a variable plotted in the graph is transformed. | 3 class periods |
| Assessment: Stopping Distances | | varies |
| **Unit III: Mathematical Models from Data** | | |
| Lesson 7: Transforming Data Using Logarithms | Recognize how transforming either scale of a graph with the logarithmic function changes the shape of the graph. | 2 class periods |
| Lesson 8: Finding an Equation for Nonlinear Data | Find the equation of a nonlinear data set using transformations. | 2 class periods |
| Lesson 9: Residuals | Use plots of residuals to help assess how well a mathematical model fits the data. | 3 class periods |
| Lesson 10: Correlation: $r$ and $r^2$ | Use the correlation coefficient and the square of the correlation coefficient along with residual plots to help assess how well a mathematical model fits a data set. | 3 class periods |

| LESSON | OBJECTIVE | PACING |
|---|---|---|
| Lesson 11: Developing a Mathematical Model | Use the knowledge of transformations, logarithms, residuals, and correlation to develop a mathematical model. | 3 class periods |
| Project: Alligators' Lengths and Weights | | varies |
| Assessment: The Growth of Bluegills | | varies |
| | | approximately 5 weeks total time |

## Technology

The amount of technology that students use may vary. Graphing calculators and/or computers with graphing software are nearly essential. The pacing guide assumes the use of technology. A graphing calculator resource section, entitled *Procedures for Using the TI-83*, is included at the end of this module.

## Grade Level/Course

Grades 10–12, Algebra II or precalculus in connection with a unit on exponential and logarithmic functions

### Use of Data Sets and Teacher Resources

The data sets listed below are on the IBM disk and the Macintosh disk that accompany this Teacher's Edition. The Resource Materials are referenced in the Materials section at the beginning of the lesson commentary.

| LESSON | DATA SETS | RESOURCES |
|---|---|---|
| **Unit I: Patterns and Scale Changes** | | |
| Lesson 1: Patterns | Length and Width | |
| Lesson 2: Changes in Units on the Axes | Year and Deficit<br>Year and Population<br>Answers to Questions 1, 7, and 16 | |
| Assessment: Speed Versus Stopping Distance and Height Versus Weight | Speed and Reaction Distance<br>Women's Height and Weight<br>Men's Height and Weight | *Unit I Test* |
| **Unit II: Functions and Transformations** | | |
| Lesson 3: Functions | | |
| Lesson 4: Patterns in Graphs | Year and Particles in Air<br>Year and Silver Production<br>Year and Numbers of Stamps | |
| Lesson 5: Transforming Data | Sample Sound and Decibels<br>Answer to Question 1 | *Activity Sheet 1*, (Questions 8 and 9) |
| Lesson 6: Exploring Changes on Graphs | Ball, Circumference, and Volume | *Activity Sheets 2–4* (Questions 1–8) |
| Assessment: Stopping Distances | Speed and Total Stopping Distance<br>Year and Motor-Vehicle Registration | *Unit II Test* |
| **Unit III: Mathematical Models from Data** | | |
| Lesson 7: Transforming Data Using Logarithms | | |
| Lesson 8: Finding an Equation for Nonlinear Data | Time and Height of Bounce for Ball<br>Distance and Time for Metal Ball | |
| Lesson 9: Residuals | Year and Population, 1790–1980<br>Median Income of Men and Women | |
| Lesson 10: Correlation: $r$ and $r^2$ | Study Hours and Grade-Point Average<br>Crime Rate and Per-Capita Spending | |
| Lesson 11: Developing a Mathematical Model | Chestnut Oak Trees<br>Age and Population Afraid of Flying | |
| Project: Alligators' Lengths and Weights | Alligators' Lengths and Weights | |
| Assessment: The Growth of Bluegills | Bluegills' Lengths | *Unit III Test* |

UNIT I

# Patterns and Scale Changes

## LESSON 1

# Patterns

**Materials:** graph paper, rulers
**Technology:** graphing calculators or computer (optional)
**Pacing:** 2 class periods

### Overview

There are three methods that mathematicians and statisticians use to find, examine, and represent data patterns: numeric, geometric (graphs or pictures), and symbolic (formulas). Each representation yields different information and aids in understanding. Studying the mathematical properties of patterns helps students make sense out of data. In this lesson, students will study the information available when patterns are presented as collections of numbers, graphs, and symbolic representations.

### Teaching Notes

It is important to allow students time to explore patterns to help them recognize that many different patterns can exist in one data set. Students are prone to find one pattern and be satisfied with that discovery. One of the purposes of this lesson is to encourage students to find as many patterns as they can within a data set. In doing so, they should become aware that there may be various ways to represent (model) a given data set. The connections among the three representations of data is very important. They must be encouraged to represent their pattern in each of the three ways. The real-world contexts provides a familiar setting in which to discover some not-so-obvious applications of mathematics.

### Follow-Up

Patterns are all around. Have the students discuss patterns they come in contact with in their daily lives and classify the patterns as numeric, geometric, or symbolic. Encourage them to describe those familiar patterns in other ways and discuss what further information each representation adds to their understanding of a real-life situation. You might provide them with a geometric pattern and ask them to describe that pattern numerically and symbolically if they do not identify any from their world.

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LESSON 1

# Patterns

**What information can be learned about a pattern expressed numerically?**

**What information can be learned about a pattern expressed geometrically with a graph or picture?**

**What information can be learned about a pattern expressed symbolically?**

Recognition of patterns is an integral part of the work done by mathematicians and statisticians. Patterns are all around us, and you need to develop an ability to find them in shapes, symbols, and data. This unit will reinforce finding and representing patterns in the world around us.

**OBJECTIVE**

Understand how ordered pairs, graphs, equations, and tables can be used to describe patterns.

*"Spaceship Earth Epcot,"* by William Means

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## Solution Key

### Discussion and Practice

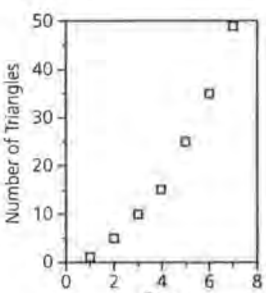
Note: Students may have difficulty relating the picture pattern to the numeric pattern. You may want to give more examples and a brief explanation to clarify.

**1.** **a.** The first member of the ordered pair is the row number and the second is the total number of triangles up to and including that row.

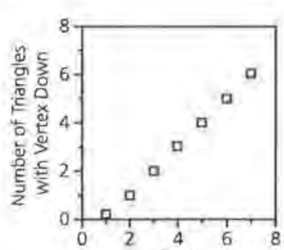
**b.** The first number is the row number and the second is the number of triangles in that row whose vertex angle is directed downward.

**2.** Answers will vary. (1, 1), (2, 2), (3, 3) ... (row number, number of triangles with vertex directed upward); (1, 3), (2, 7), (3, 11), (4, 15)..... (row number, total number of unit segments in that row)

**3.** Explanations will vary. For example:



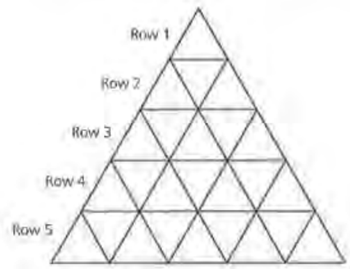

As the first number increases, the second number also increases and the graph is curved.




As the first number increases, the second number also increases. The graph appears to be straight.

**INVESTIGATE**

At Epcot Center in Orlando, Florida, *Spaceship Earth*, a 180-foot high geosphere, was constructed with triangles. Can you think of other places you may have seen triangle patterns? Consider the triangle pattern below. How many triangles do you see?




Some relationships can be modeled and studied further. When you model information, it may be helpful to first write it with numbers or symbols. For example, ordered pairs can be used to represent patterns.

**Discussion and Practice**

In the triangle pattern above, the visual pattern of the row and number of triangles in that row, (row, triangles), can be represented by the ordered pairs (1, 1), (2, 3), (3, 5), (4, 7), . . . . In symbols this can be written $(r, t)$.

**1.** Examine each set of ordered pairs below. In each example the ordered pairs represent a pattern in the triangle above. Explain how they relate to a visual pattern in the triangle.

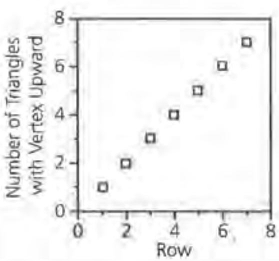
**a.** (1, 1), (2, 4), (3, 9), (4, 16), . . .

**b.** (1, 0), (2, 1), (3, 2), (4, 3), . . .

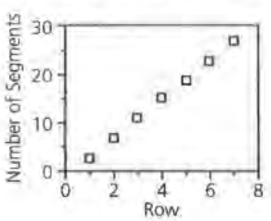
**2.** Write two other ordered-pair patterns that can be generated from the triangle picture. Explain how the numbers in your ordered pairs are related.

**3.** Graph each set of ordered pairs in Question 1. Write a sentence to describe the patterns you see in the graph. As the first number in the ordered pair increases, what happens to the second number?

4. Explanations will vary. For example:




As the first number increases, the second number also increases. The graph appears to be straight.

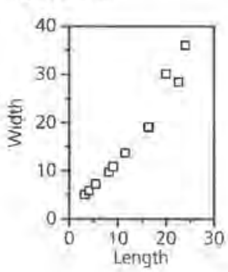



As the first number increases, the second number also increases. The graph appears to be straight.

5. The equation gives you the value for $t$ when the value for $r$ is given.

6. Sample: The equation for the total number of triangles ($T$) given the row number ($r$) would be $T = r^2$.

7. Students should examine the data set and plot it on paper. The linear relationship becomes apparent when the ordered pairs are graphed.




# STUDENT PAGE 5

4. Graph the ordered pairs you found in Question 2. Write a sentence to describe the patterns you see in the graph. As the first number in the ordered pair increases, what happens to the second number?

An equation may be used to describe the relationship between the first and second number in an ordered pair. Recall that for the ordered pairs ($r$, $t$) in the triangle pattern on page 4, $r$ = row number and $t$ = number of triangles in that row.

5. How do the ordered pairs (1, 1), (2, 3), (3, 5), (4, 7), . . . relate to the equation $t = 2r - 1$, where $r$ represents the first number in the ordered pair and $t$ represents the second number in the ordered pair?

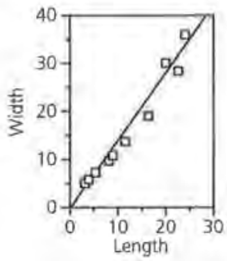
6. Write an equation to describe the pattern in at least one of the other sets of ordered pairs in Questions 1 and 2.

7. These data relate to common objects that you probably have in your home. They were collected at a department store. Each row in this table can be considered an ordered pair. Plot the following ordered pairs (length, width) and describe the pattern.

| Length | Width |
|---|---|
| 8 | 10 |
| 5 | 7 |
| 4 | 6 |
| 22 | 28 |
| 3.5 | 5 |
| 8.5 | 11 |
| 11 | 14 |
| 16 | 20 |
| 20 | 30 |
| 24 | 36 |

Recall from your previous work that lines such as least squares or median-fit lines are used to show the linear trend of a graph. These lines are often used to determine values *between* those given on a table and *beyond* the values given on a table. In most cases, the straight line will not pass through all the points on the graph but is used to summarize the linear relation between the variables, just as mean or median is used to summarize the center of a univariate set of data. Equations of straight lines can be quickly determined from ordered pairs and then be used to make predictions.

**8.**




**a.** Answers will vary. (7, 9.3), (10, 13.5), (15, 20.6)

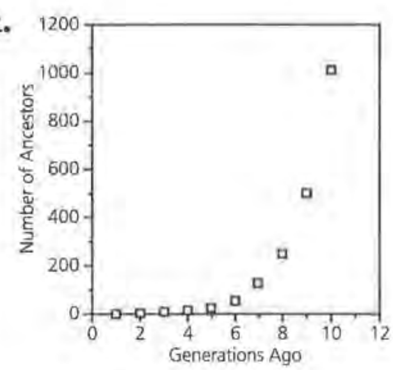
**b.** Answers will vary. $W = 1.42L - 0.68$, or $W = 1.5L$

**9.** Answers will vary. Sample: The data could be the dimensions of picture frames or sizes of pieces of paper or window sizes. One could use inches or centimeters.

**10.**

| Generations Ago | Number of Ancestors |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |
| 8 | 256 |
| 9 | 512 |
| 10 | 1024 |

**11.**




**12.** As the first number increases, the second number increases. The graph appears to be a curved line.

# STUDENT PAGE 6

**8.** Draw a line on your graph.

**a.** Use the line drawn to determine three ordered pairs that could have also been in the data.

**b.** Write an equation for your line.

**9.** What do you think the data on the table in Question 7 represent? What might be the appropriate units for these data?

**Example: Ancestor Patterns**

In Salt Lake City, Utah, there is a genealogy library that helps people searching for information about their ancestors. Books containing information such as birth, death, and immigration records sometimes make it possible to locate the names of ancestors who lived several hundred years ago. The number of ancestors you have in past generations forms a mathematical pattern. For example, you have 2 parents and 4 grandparents.

**10.** Write the information for 10 generations in a table like this.

| Generations Ago | Number of Ancestors |
|---|---|
| 1 | 2 |
| 2 | ____ |
| 3 | ____ |
| 4 | ____ |
| 5 | ____ |
| 6 | ____ |
| 7 | ____ |
| 8 | ____ |
| 9 | ____ |
| 10 | ____ |

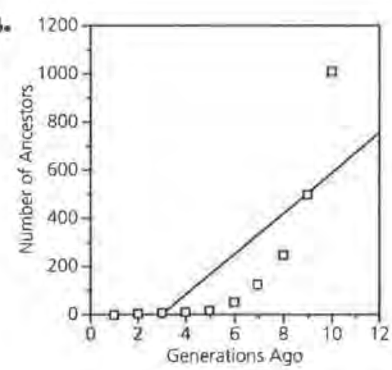
**11.** Make a scatter plot of the ordered pairs.

**12.** Write a few sentences to describe the patterns on the graph.

**13.** Draw a straight line through your scatter plot that appears to come closest to all of the data points, and use it to make some predictions.

**a.** How do this graph and its line compare to the data set and line in Question 8?

**13.**




Answers will vary. Predictions from the line: (5, 161), (12, 770)

**a.** This line does not fit the data set very well.

(13) b. The linear equation would not be a very good summary because the ordered pairs generated by that equation would not be anywhere near the values of the given data set.

14. a. $A = 2^G$

b. $A = 2^{12} = 4096$

c. $33{,}554{,}432/2 = 16{,}777{,}216$
$33{,}554{,}432 \cdot 2 = 67{,}108{,}864$

The number of ancestors is a power of 2. To determine the number of ancestors in one less generation, divide by 2. To determine the number of ancestors in the next generation, multiply by 2.

## Practice and Applications

15. Graph, list, table, ordered pairs, and equation

16. Nonlinear, linear, concave up, concave down, sloping up, sloping down, increasing, decreasing, and so on

STUDENT PAGE 7

b. Use your predictions to determine if a linear equation would be a good summary of the pattern. Explain why or why not.

14. Study the relationship between the first and second variables in each of your ordered pairs from the table in Question 10.

a. Write an equation that can be used to describe the relationship.

b. Determine how many ancestors you had 12 generations ago.

c. Suppose you had 33,554,432 ancestors 25 generations ago. How many did you have 24 generations ago? 26 generations ago? Explain how you determined your answers.

**Summary**

Studying the mathematical properties of patterns helps you make sense out of data. In this module, you will continue to study patterns and their graphs. Notice that some graph patterns are straight and some are curved. All of the data points in a set do not have to lie exactly on a line for the trend to be considered a straight line. A linear equation is used to model straight-line trends. When data follow curved patterns, equations that are not linear may be used to describe their trends.

**Practice and Applications**

15. List at least two different ways mathematics can be used to show a pattern.

16. Write at least three different words or phrases that can be used to describe trends on the graphs you made.

## LESSON 2

# Changes in Units on the Axes

**Materials:** graph paper, rulers, Unit I Quiz
**Technology:** graphing calculators or computer (optional)
**Pacing:** 2 class periods

### Overview

Paying attention to the units attached to a number is important when interpreting data. If the units of a data set are changed, the scale changes. This could affect the interpretation of the pattern. For example, changing the units from centimeters to meters in a data set changes the appearance of the pattern or graph and can influence the message the data set conveys. This lesson enables a student to discover the effect, if any, a change in scale would have on the numeric, geometric, and symbolic representation of the data.

### Teaching Notes

Students must be afforded the opportunity to discover just how much influence the unit has upon the interpretation of the information presented in the table or graph. A unit change or scale change, such as a change from an amount of money in number of nickels to an amount of money in number of quarters, does not affect the value of the amount. That unit or scale change would affect numeric representations, graphs, and the symbols used to represent numbers. Encourage students to use their knowledge of the various kinds of units and scales to conjecture and then discover what effect applying changes in units might precipitate.

### Follow-Up

Students could bring some data with which they are familiar and change the units or scales to see what effect these changes might cause. Have them analyze graphs and tables of values in the daily paper and discuss whether or not they think the choice of scale or units influences the information or impression one gets from looking at the representation.

STUDENT PAGE 8

## Solution Key

### Discussion and Practice

1. These data should be collected from students in the class. An example is provided below.

LESSON 2

# Changes in Units on the Axes

**What effect will the change of unit or scale have on the numeric representation?**

**What effect will the change of unit or scale have on the geometric representation?**

**What effect will the change of unit or scale have on the symbolic representation?**

INVESTIGATE

Often the effect of changing the units of measure for the items being graphed or being represented in a table is completely overlooked. For instance, if you wanted to conduct a survey to determine about how much loose change people carry in their pockets or purses, what units could you use?

OBJECTIVE

Understand how changes in units affect tables and graphs.

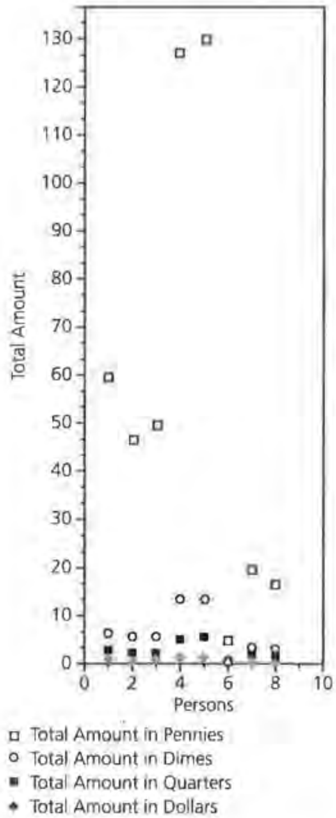
**Discussion and Practice**

Collect information from students in class to answer the question "How much loose change are you carrying?"

1. Record the information in a table like the one shown on the next page. Show each amount four different ways, expressing answers in decimal form.

| Person | Total Amount Expressed in Number of Pennies | Total Amount Expressed in Number of Dimes | Total Amount Expressed in Number of Quarters | Total Amount Expressed in Number of Dollars |
|---|---|---|---|---|
| Example | 57 | 5.7 | 2.28 | 0.57 |
| 1 | 61 | 6.1 | 2.44 | 0.61 |
| 2 | 47 | 4.7 | 1.88 | 0.47 |
| 3 | 50 | 5 | 2.00 | 0.50 |
| 4 | 128 | 12.8 | 5.12 | 1.28 |
| 5 | 130 | 13 | 5.2 | 1.30 |
| 6 | 5 | 0.5 | 0.2 | 0.05 |
| 7 | 22 | 2.2 | 0.88 | 0.22 |
| 8 | 15 | 1.5 | 0.6 | 0.15 |

2. Scaling for these four scatter plots on one graph will be difficult. Students need the experience to appreciate the difficulty involved in choosing appropriate scales and recognizing the difference a scale change makes.




3. Answers will vary because students will have original data. Most frequently, the median would be the better choice. If the data did not contain any outliers, the mean would be just as good.

4. $q = 0.4d$ or $d = \frac{q}{0.4}$

5. Answers will vary. Each scale is some multiple of any of the others. Any one of the scales could be chosen because the graphs are all similar and provide the same information. Changing the scale only makes the graphs appear different. The values are the same.

## STUDENT PAGE 9

| Person | Total Amount Expressed in Number of Pennies | Total Amount Expressed in Number of Dimes | Total Amount Expressed in Number of Quarters* | Total Amount Expressed in Number of Dollars |
|---|---|---|---|---|
| Example | 57 | 5.7 | 2.28 | 0.57 |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| • | | | | |
| • | | | | |

*To express the amount in quarters, divide the number of cents by 25.

2. Make four scatter plots on one coordinate plane using the ordered pairs (person, total amount) for each amount.
   a. Money people carry expressed in number of pennies
   b. Money people carry expressed in number of dimes
   c. Money people carry expressed in number of quarters
   d. Money people carry expressed in number of dollars
3. Find the mean and the median for each column. Would the mean or the median better describe how to represent the typical amount of change a person in the class has? Explain why you made that choice.
4. Use mathematical symbols to describe the relationship between the amount in number of quarters and the amount in number of dimes.
5. Write a summary paragraph to explain the relationship between any two of the units used in Question 4. Discuss why either unit could be used.

6. a. 31,700 years
b. $4.5 \times 10^6$ = 4,500,000 inches or approximately 71 miles

# STUDENT PAGE 10

The federal debt is the amount of money the federal government owes. Most of it is owed to citizens who lend the government money by buying bonds or treasury bills. The debt increased steadily from 1980–1995 because of deficit spending and increases in interest owed on the debt. Deficit spending occurs when the government spends more in a year than it takes in through taxes and other revenues. The federal deficit is added to the federal debt each year.

6. The federal debt is usually described in trillion dollars, written as $1,000,000,000,000, with $1 as the unit. It is hard to grasp how much one trillion dollars is.
   a. A billion seconds is 31.7 years. How long is a trillion seconds?
   b. If a million dollars in $1,000 bills would make a stack four and one-half inches high, how high would a trillion dollars in the same currency stack?

How much is the federal debt increasing each year? Study these data to help you answer the questions.

| Year | Federal Debt | Federal Debt (dollars) | Federal Debt (billion dollars) |
|---|---|---|---|
| 1980 | 0.9091 trillion dollars | ______ | ______ |
| 1981 | 0.9949 trillion dollars | ______ | ______ |
| 1982 | 1.137 trillion dollars | ______ | ______ |
| 1983 | 1.372 trillion dollars | ______ | ______ |
| 1984 | 1.565 trillion dollars | ______ | ______ |
| 1985 | 1.818 trillion dollars | ______ | ______ |
| 1986 | 2.121 trillion dollars | ______ | ______ |
| 1987 | 2.346 trillion dollars | 2,346,000,000,000 | 2,346 |
| 1988 | 2.601 trillion dollars | ______ | ______ |
| 1989 | 2.868 trillion dollars | ______ | ______ |
| 1990 | 3.207 trillion dollars | ______ | ______ |
| 1991 | 3.598 trillion dollars | ______ | ______ |
| 1992 | 4.002 trillion dollars | ______ | ______ |
| 1993 | 4.351 trillion dollars | ______ | ______ |
| 1994 | 4.644 trillion dollars | ______ | ______ |
| 1995 | 4.921 trillion dollars | ______ | ______ |

Source: U.S. Treasury Department

STUDENT PAGE 11

7. Rewrite the numbers in the "Federal Debt" columns in dollars and billion dollars as shown in the row for 1987.

8. The following is a graph with the years on the x-axis and the debt in dollars on the y-axis. Make a conjecture as to what the graph might look like if the debt in the trillion-dollars column had been plotted on the y-axis.


9. Plot (year, federal debt in billion dollars). Label the axes. Describe in a short paragraph what, if any, changes occur in the graph when the units on the y-axis are changed from dollars to billion dollars.

10. The population of the United States in 1980 was about 230,000 thousand people. What was the amount of the federal debt per thousand people in 1980? What was the dollar amount of the federal debt per person in 1980?

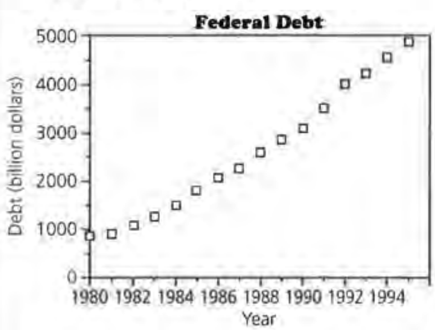
11. Our taxes support the federal government. When the federal government spends more money than it receives in taxes and other revenues, it is called "deficit spending," that is, spending money the government does not have. This amount, referred to as the "deficit," must be borrowed and is added to the federal debt each year.

7.

| Year | Federal Debt ($) | Federal Debt (billion $) |
|---|---|---|
| 1980 | 909,100,000,000 | 909.1 |
| 1981 | 994,900,000,000 | 994.9 |
| 1982 | 1,137,000,000,000 | 1,137 |
| 1983 | 1,372,000,000,000 | 1,372 |
| 1984 | 1,565,000,000,000 | 1,565 |
| 1985 | 1,818,000,000,000 | 1,818 |
| 1986 | 2,121,000,000,000 | 2,121 |
| 1987 | 2,346,000,000,000 | 2,346 |
| 1988 | 2,601,000,000,000 | 2,601 |
| 1989 | 2,868,000,000,000 | 2,868 |
| 1990 | 3,207,000,000,000 | 3,207 |
| 1991 | 3,598,000,000,000 | 3,598 |
| 1992 | 4,002,000,000,000 | 4,002 |
| 1993 | 4,351,000,000,000 | 4,351 |
| 1994 | 4,644,000,000,000 | 4,644 |
| 1995 | 4,921,000,000,000 | 4,921 |

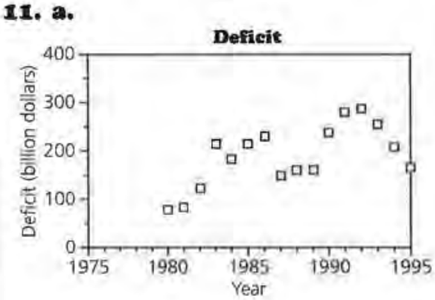
8. Answers will vary. If the graph had been drawn with the debt in trillion dollars on the y-axis, the only recognizable change would have been the scale on the y-axis.

9. The only change is the number scale on the y-axis.




10. $909,100,000,000/230,000 = $3,952,608.70 per thousand people
$909,100,000,000/230,000,000 = $3,952.61 per person

**11. a.**




**b.** The federal debt is constantly increasing, while the federal deficit has more ups and downs. The federal debt has an overall increasing trend.

## Practice and Applications

**12.** Answers will vary; however, the students would most likely choose dollars as it would be the least cumbersome.

**13.** It is more obvious that the federal debt is increasing at an increasing rate.

**14.** The size of the debt is not affected by the units chosen. Answers will vary, but many would prefer the unit involving the fewest zeros.

STUDENT PAGE 12

| Year | Deficit (billion dollars) |
|---|---|
| 1980 | 73.4 |
| 1981 | 79.3 |
| 1982 | 128.5 |
| 1983 | 208.7 |
| 1984 | 186.8 |
| 1985 | 213.3 |
| 1986 | 223.1 |
| 1987 | 152.0 |
| 1988 | 153.6 |
| 1989 | 149.9 |
| 1990 | 221.7 |
| 1991 | 269.5 |
| 1992 | 288.7 |
| 1993 | 252.5 |
| 1994 | 205.4 |
| 1995 | 165.5 |

Source: U.S. Government Printing Office

**a.** Make a scatter plot of (year, deficit in billion dollars).

**b.** Compare your Federal Deficit and Federal Debt graphs. Identify the similarities.

### Summary

Paying attention to the units attached to a number is important when interpreting data. A unit change or scale change, such as from an amount of money in number of nickels to an amount of money in number of quarters, does not affect the value of the amount. However, a unit change or scale change may affect numeric representations, graphs, and the symbols used to represent numbers.

### Practice and Applications

**12.** If you had to choose a single unit to represent the amount of loose change a person carries, would you prefer to use pennies, dimes, quarters, or dollars? Write an argument supporting your choice.

**13.** What additional information can you gain from a graph of the federal debt that might not be apparent in the table?

**14.** Is the size of the debt affected by the units? What units do you prefer to use to describe the federal debt? Explain why.

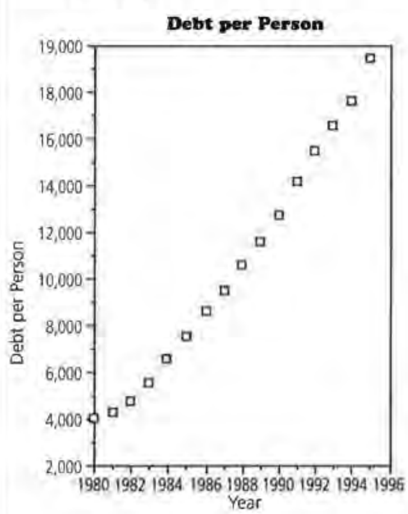
**15. a.** $165,500,000,000/262,755,000 = $629.86 per person

**b.** $629.86/400 ≈ 1.575 weeks ≈ 7.875 days

**16.**

| Year | Debt per Person |
|---|---|
| 1980 | $4,012.87 |
| 1981 | $4,335.72 |
| 1982 | $4,907.97 |
| 1983 | $5,868.46 |
| 1984 | $6,636.28 |
| 1985 | $7,641.10 |
| 1986 | $8,832.61 |
| 1987 | $9,682.65 |
| 1988 | $10,638.08 |
| 1989 | $11,619.85 |
| 1990 | $12,894.12 |
| 1991 | $14,269.96 |
| 1992 | $15,691.72 |
| 1993 | $16,877.42 |
| 1994 | $17,837.53 |
| 1995 | $18,728.47 |

**a.** The plot is a curve, concave upward, nonlinear.




**b.** Answers will vary. Using the graph in part a, the debt per person in 1996 would be about $19,600.

## STUDENT PAGE 13

**15.** This table contains information about the United States population.

| Year | Population (thousands) |
|---|---|
| 1980 | 226,546 |
| 1981 | 229,466 |
| 1982 | 231,664 |
| 1983 | 233,792 |
| 1984 | 235,875 |
| 1985 | 237,924 |
| 1986 | 240,133 |
| 1987 | 242,289 |
| 1988 | 244,499 |
| 1989 | 246,819 |
| 1990 | 248,718 |
| 1991 | 252,138 |
| 1992 | 255,039 |
| 1993 | 257,800 |
| 1994 | 260,350 |
| 1995 | 262,755 |

Source: *Statistical Abstract of the United States*, 1997

**a.** How much was the federal deficit in dollars per person in 1995?

**b.** Working at $10.00 per hour, 40 hours per week, how many weeks would it take you to pay your share of the deficit in 1995? How many days?

**16.** Create a table that contains the amount of federal debt per person for the years 1980–1995.

**a.** Make a scatter plot of the debt-per-person data for the years 1980–1995 and describe its shape.

**b.** Use your graph to estimate the federal debt per person for 1996.

**c.** Is it reasonable to calculate the debt per state? Explain.

**c.** No; it doesn't have any meaning, since the populations of states vary.

STUDENT PAGE 14

ASSESSMENT

# Speed Versus Stopping Distance and Height Versus Weight

In driver-education classes, students are usually taught to allow, under normal driving conditions, one car length for every ten miles of speed and more distance in adverse weather or road conditions. The faster a car is traveling, the longer it takes the driver to stop the car. The stopping distance (the total distance required to bring an automobile to a complete stop) depends on the driver-reaction distance (the distance traveled between deciding to stop and actually engaging the brake) and the braking distance (the distance required to bring the automobile to a complete stop once the brake has been applied). The data below represent average driver-reaction distances based on tests conducted by the U.S. Bureau of Public Roads. Average total stopping distances will be investigated later in the module.

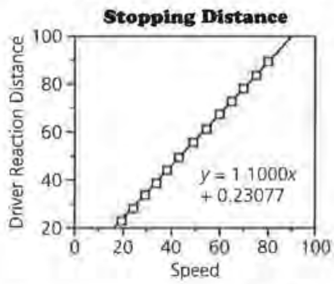
| Speed (mph) | Driver-Reaction Distance (ft) |
|---|---|
| 20 | 22 |
| 25 | 28 |
| 30 | 33 |
| 35 | 39 |
| 40 | 44 |
| 45 | 50 |
| 50 | 55 |
| 55 | 61 |
| 60 | 66 |
| 65 | 72 |
| 70 | 77 |
| 75 | 83 |
| 80 | 88 |

Source: U.S. Bureau of Public Roads

## Solution Key

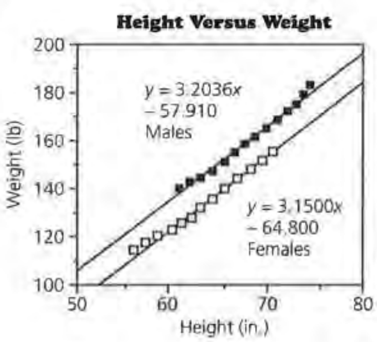
1. The greater the speed the longer the distance required for a driver to react. The rate of change appears constant.

2. A straight line can be used to describe this graph because of the constant differences between the $y$-values for consecutive $x$-values. The equation is $y = 1.1x + 0.23077$.




3. Answers will vary. (27, 29.9) (32, 35.4), (63, 69.5) The predictions will be very accurate since the plot is a straight line.

4.




5. On the average, for every inch of growth in males, the weight will increase approximately 3.2 pounds. On the average, for every inch of growth in females, the weight will increase 3.15 pounds.

6. The federal deficit per person in 1992 was 11.3198 hundreds of dollars. The federal deficit per person in 1992 was 1.13198 thousands of dollars.

# STUDENT PAGE 15

1. Describe any pattern in the data table using the knowledge you gained in this unit.

2. Make a scatter plot of (speed, reaction distance). Can a straight line be used to summarize the trend of the graph? Explain. Draw a line and find its equation.

3. Use the equation to make at least 3 predictions for reaction distances at speeds not included in the table. How accurate do you think your predictions are?

The following data sets come from the Mayo Clinic Family Healthbook relating average height to average weight for both males and females.

| Women's Height (inches) | Women's Weight (pounds) |
|---|---|
| 57 | 117 |
| 58 | 119 |
| 59 | 121 |
| 60 | 123 |
| 61 | 126 |
| 62 | 129 |
| 63 | 133 |
| 64 | 136 |
| 65 | 140 |
| 66 | 143 |
| 67 | 147 |
| 68 | 150 |
| 69 | 153 |
| 70 | 156 |
| 71 | 159 |

Source: *Mayo Clinic Family Healthbook*

| Men's Height (inches) | Men's Weight (pounds) |
|---|---|
| 61 | 139 |
| 62 | 142 |
| 63 | 144 |
| 64 | 147 |
| 65 | 150 |
| 66 | 153 |
| 67 | 156 |
| 68 | 159 |
| 69 | 162 |
| 70 | 165 |
| 71 | 169 |
| 72 | 172 |
| 73 | 176 |
| 74 | 180 |
| 75 | 185 |

Source: *Mayo Clinic Family Healthbook*

4. Use the data to graph (women's height, women's weight) or (men's height, men's weight). Draw a straight line that seems to come closest to all the data points on your graph. Find the equation of your line.

5. What does the slope of the line tell you, and how does it relate to the table?

6. The federal deficit per person in 1992 was $1131.98. Write an estimate of this amount with hundred dollars per person as the unit. Write this amount with thousand dollars per person as the unit.

UNIT II

# Functions and Transformations

## LESSON 3

# Functions

**Materials:** graph paper, rulers
**Technology:** graphing calculators or computer
**Pacing:** 2 class periods

### Overview

In modeling, statisticians and mathematicians look for patterns that can be used to explain and/or understand a data set. The process of *mathematical modeling* consists of examining a data set for patterns and looking for a function whose properties most closely represent the data's properties. This lesson will review the properties and shapes of specific functions so that they may be used in this process. Important definitions in the study of mathematics and statistics used in mathematical modeling are *response variable, explanatory variable, relations,* and *functions.* When examining data and determining the dependence of one variable upon another, you can identify the dependent variable as the ***response variable.*** The other variable is then referred to as the ***explanatory variable. Relations*** are sets of *ordered* pairs of the two variables. Within the set of relations is a subset called "functions." ***Functions*** are relations in which every instance of the explanatory variable is paired with a single instance of the response variable. These terms will be used in the remainder of this module.

### Teaching Notes

The process of mathematical modeling used in this lesson consists of examining a data set and finding a function whose properties most closely represent its properties. In modeling, statisticians and mathematicians look for patterns that can be used to explain and/or understand a data set. There are many specific functions that are useful in mathematical modeling: *linear functions, logarithmic functions, exponential functions, power functions, quadratic functions,* and *square-root functions.* Students should be familiar enough with each of these functions and their inverses in both symbols (equations) and graphical shapes that they might be able to guess at what mathematical model might be chosen to fit a particular data set once they had seen its plot.

STUDENT PAGE 19

LESSON 3

# Functions

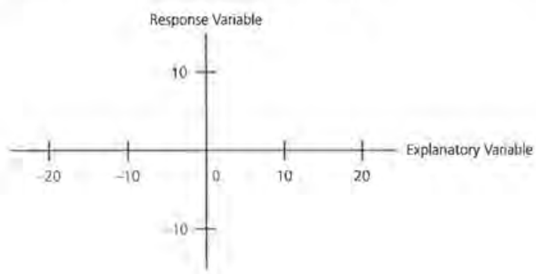
**What does the graph of each specific function type look like?**

**What is the relationship between the coefficients of a function's expression in symbolic form and the graph of that function?**

**OBJECTIVE**

Recognize graphs and equations for different functions.

In modeling, statisticians and mathematicians look for patterns that can be used to explain and/or understand a data set. The process of *mathematical modeling* consists of examining a data set for patterns and looking for a function whose properties most closely represent the data's properties. This lesson will review the properties and shapes of specific functions so that they may be used in this process. Important terms in the study of mathematics and statistics used in mathematical modeling are *response variable, explanatory variable, relations,* and *functions.* When examining data and determining the dependence of one variable upon another, you can identify the dependent variable as the ***response variable.*** The other variable is then referred to as the ***explanatory variable.***



# STUDENT PAGE 20

***Relations*** are sets of ordered pairs of the two variables. Within the set of relations is a subset called "functions." ***Functions*** are relations in which every instance of the explanatory variable is paired with a single instance of the response variable. These terms will be used throughout the remainder of this module.

## INVESTIGATE

There are many specific functions that are useful in mathematical modeling: *linear functions, logarithmic functions, exponential functions, power functions, quadratic functions, reciprocal functions,* and *square-root functions.*

It is helpful to know how the appearance of the graph of a function relates to the data it represents. For instance, what will be the appearance of a graph when the function is increasing? decreasing? constant? What information is gained about the graph of a function by knowing it has an asymptote? And how does changing the rate of change affect the appearance of the graph of a function?

### Discussion and Practice

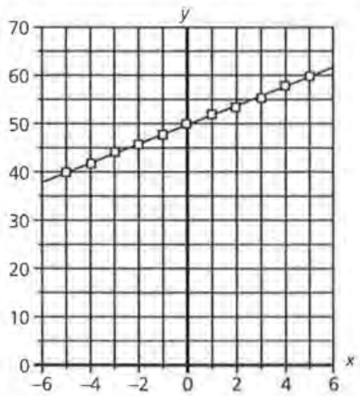
It is important for you to be able to recognize functions by their graphs and equations.

*Linear Function* A general form of the equation of a linear function is $y = bx + a$ or $y = b(x - c) + a$, where $a$, $b$, and $c$ are constants. The graph appears as a straight line. The slope is determined by the numerical value of the constant $b$ in the equation. If $b$ is positive, the line slopes up to the right; and if $b$ is negative, the line slopes down to the right.

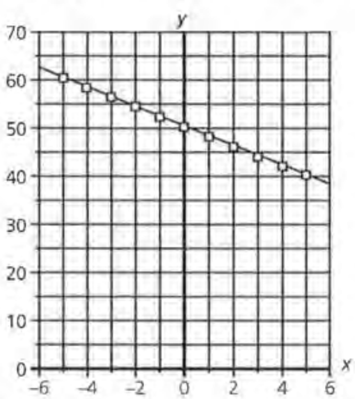
## Solution Key

## Discussion and Practice

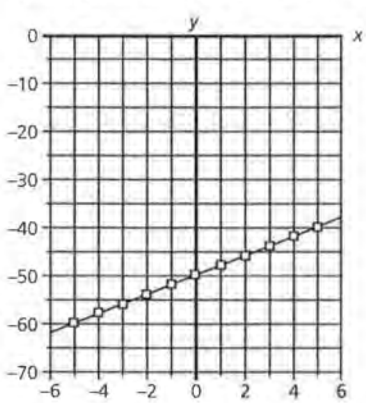
**1.** **a.** Slope, 2; sample point, (2, 54)



**b.** Slope, –2; sample point, (2, 46)

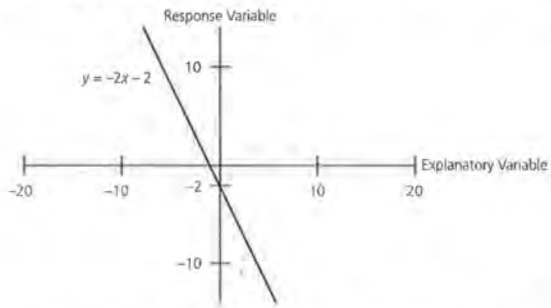


**c.** Slope, 2; sample point, (2, –46)

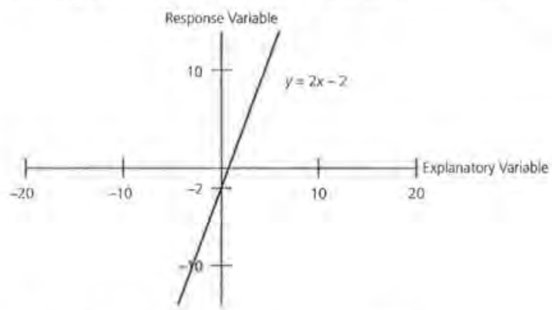


# STUDENT PAGE 21

This graph intercepts the $y$-axis at –2 and has a negative slope.




This graph intercepts the $y$-axis at –2 and has a positive slope.




**1.** For each of the following linear functions, determine the slope of the line and the coordinates of a point on the line. Then sketch the graph. You may use a graphing utility.

**a.** $y = 2x + 50$

**b.** $y = -2x + 50$

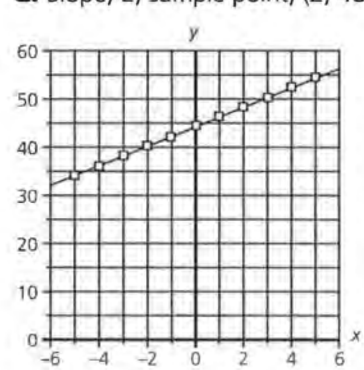
**c.** $y = 2x - 50$

**d.** $y = 2(x - 3) + 50$

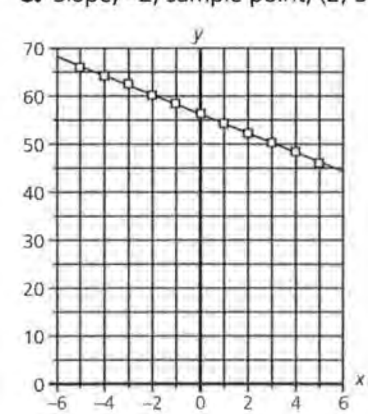
**e.** $y = -2(x - 3) + 50$

**f.** $y = 2(x - 3) - 50$

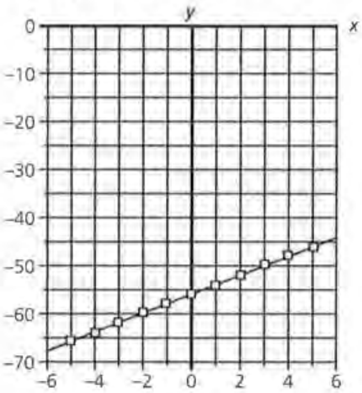
**d.** Slope, 2; sample point, (2, 48)



**e.** Slope, –2; sample point, (2, 52)

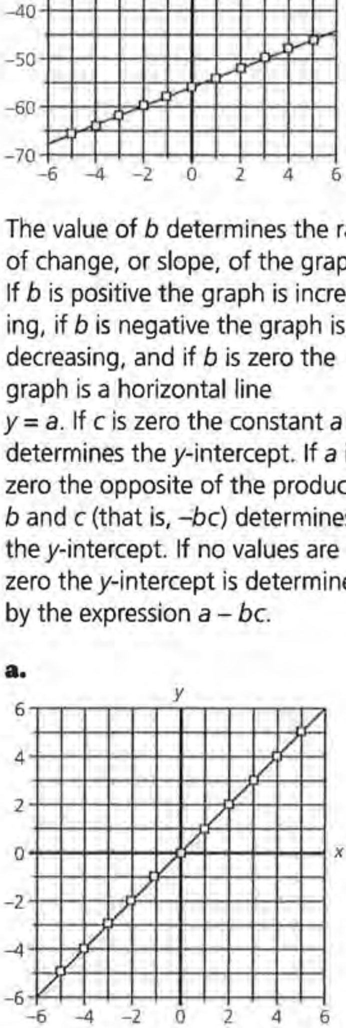


**f.** Slope, 2; sample point, (2, –52)

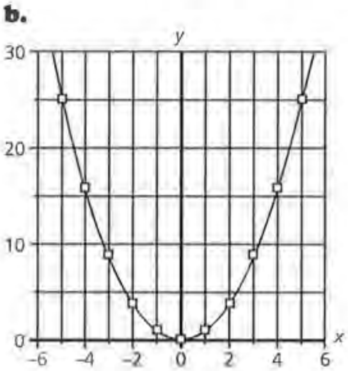


**2.** The value of $b$ determines the rate of change, or slope, of the graph. If $b$ is positive the graph is increasing, if $b$ is negative the graph is decreasing, and if $b$ is zero the graph is a horizontal line $y = a$. If $c$ is zero the constant $a$ determines the $y$-intercept. If $a$ is zero the opposite of the product of $b$ and $c$ (that is, $-bc$) determines the $y$-intercept. If no values are zero the $y$-intercept is determined by the expression $a - bc$.

**3. a.**



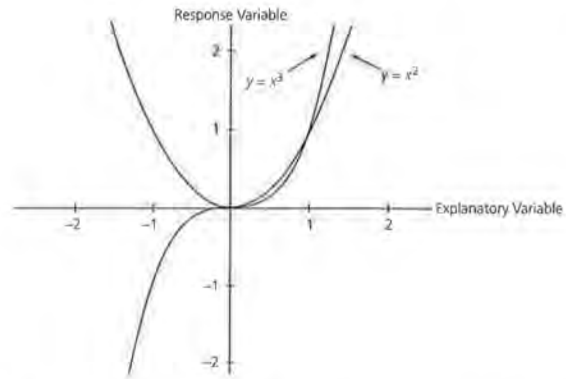
**b.**



## STUDENT PAGE 22

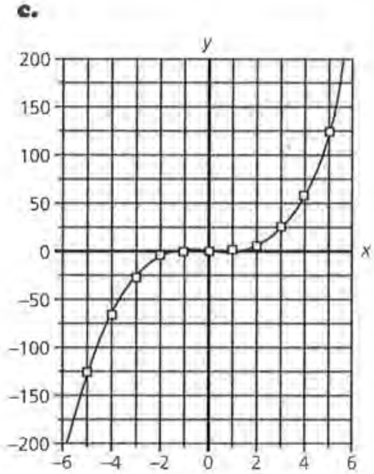
**2.** Describe the effect each of the constants $a$, $b$, and $c$ has on the graph of the equation $y = b(x - c) + a$.

*Power Function* The general form of the equation of a power function is $y = ax^b$ where $a$ and $b$ are constants. The graph appears as a smooth curve with the amount, and sometimes the direction, of the curvature influenced by the value of $b$.

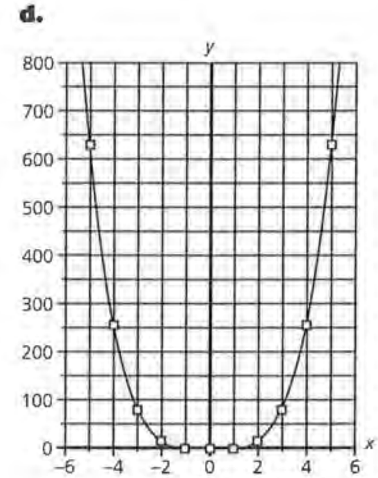



The graph above represents two power functions with positive exponents. The next two graphs represent two power functions with negative exponents. In contrast to the power functions with positive exponents, the power functions with negative exponents are decreasing functions for positive values of the explanatory variable. For negative values of the explanatory variable, the power functions with even negative exponents are increasing and those with odd negative exponents are decreasing. The rate of increase or decrease is related to the absolute value of the exponent.

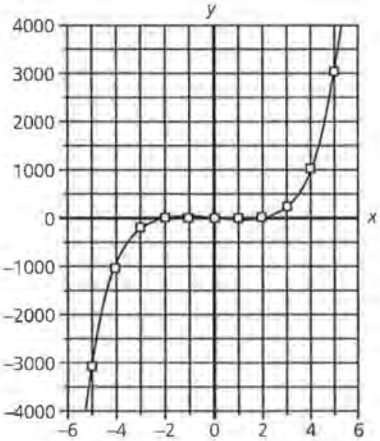
**c.**



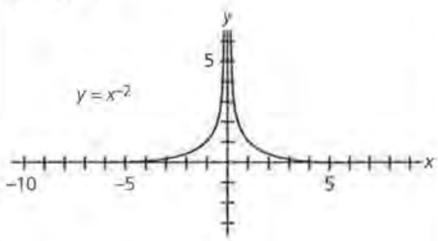
**d.**



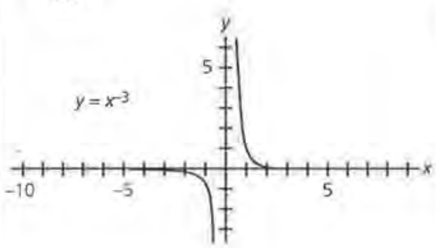
**(3) e.**



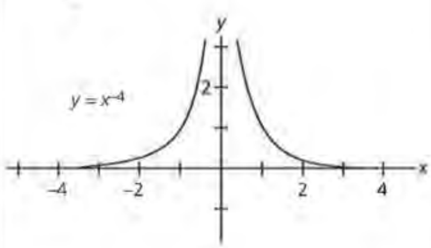
**4. a.**




**b.**



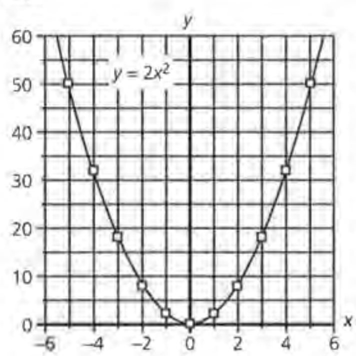

**c.**



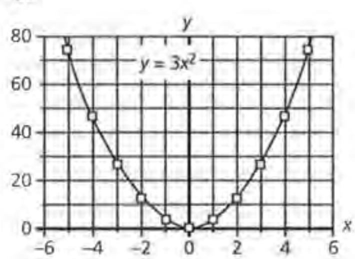

**5.** When the value of the constant $b$ is negative and odd, the graph is a decreasing function on its entire domain and has the vertical and horizontal axes as asymptotes. When the value of the constant $b$ is negative and even, the graph is an increasing function over the first half of its domain and decreasing over the second half. The two axes are again asymptotes. When the value of $b$ is positive and odd, the graph is increasing on its entire domain, from the third to the first quadrants. When $b$ is positive and even, the graph decreases in quadrant II and increases in quadrant I.

**6.** Answers will vary. Sample conjecture: The absolute value of the constant $a$ affects how wide or narrow the graph is; the sign of $a$ affects its direction.

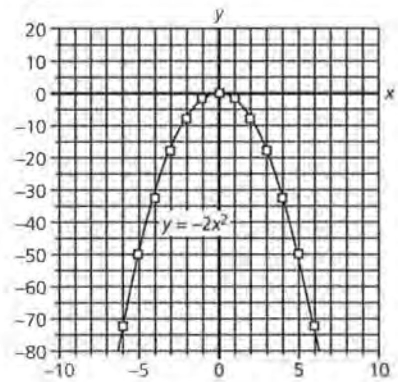
**a.**



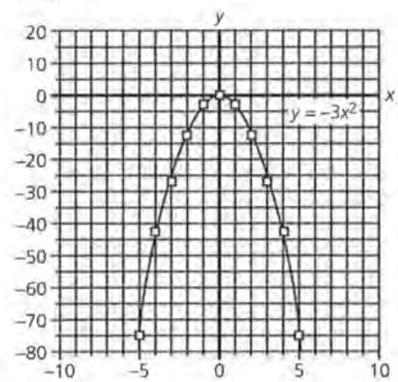

**b.**



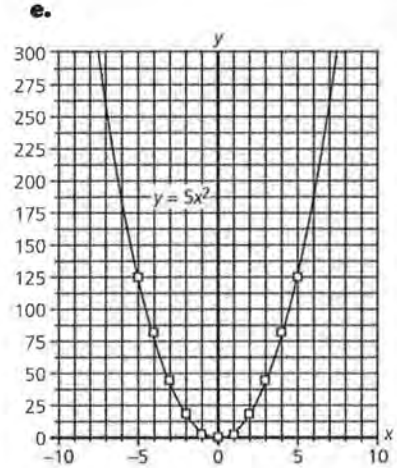

**c.**



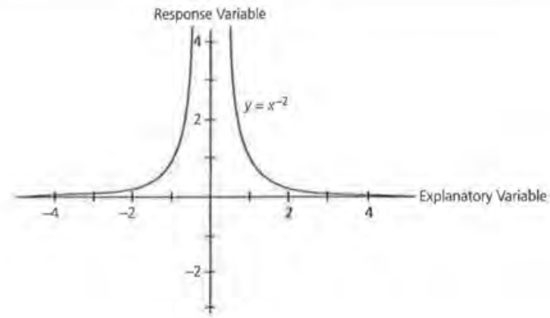

**d.**

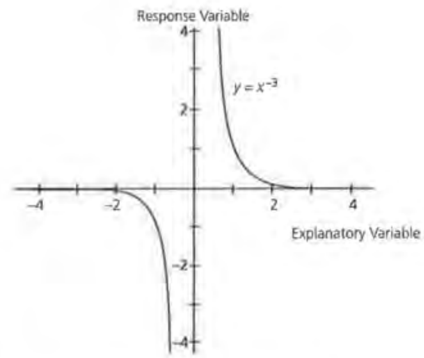



**e.**



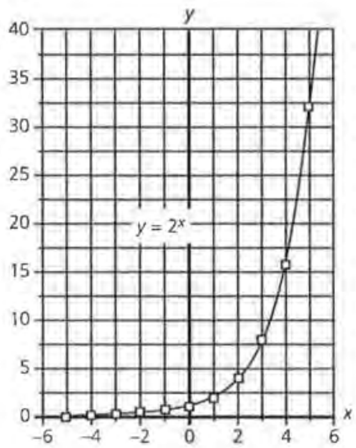
## STUDENT PAGE 23



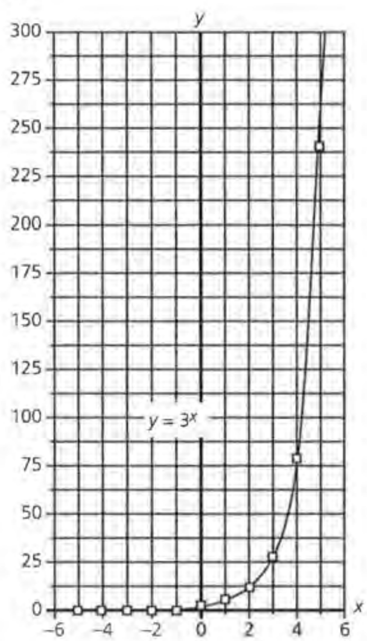




3. Sketch a graph of each function. You may use a graphing utility.
   a. $y = x^1$
   b. $y = x^2$
   c. $y = x^3$
   d. $y = x^4$
   e. $y = x^5$
4. Sketch a graph of each function. You may use a graphing utility.
   a. $y = x^{-2}$
   b. $y = x^{-3}$
   c. $y = x^{-4}$
5. Describe the general effect the constant $b$ has on the graph $y = x^b$. Let the constant $b$ take on both positive and negative values.

7. a.




b.




## STUDENT PAGE 24

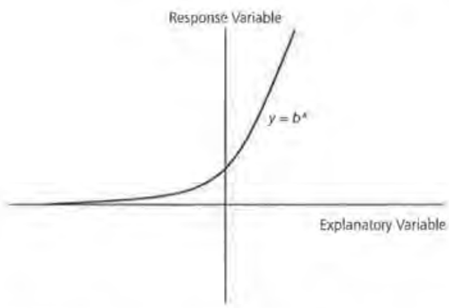
6. Make a conjecture about the effect the constant $a$ has on the graph of $y = ax^b$. With your graphing utility, graph the following to test your conjecture.

a. $y = 2x^2$ b. $y = 3x^2$

c. $y = -2x^2$ d. $y = -3x^2$

e. $y = 5x^2$

*Exponential Function* The general form of the equation of an exponential function is $y = ab^x$, where $a$ and $b$ are constants and $b > 0$ and $b \neq 1$. The graph appears as a smooth curve that increases or decreases and has a horizontal asymptote as the value of the explanatory variable approaches negative infinity.



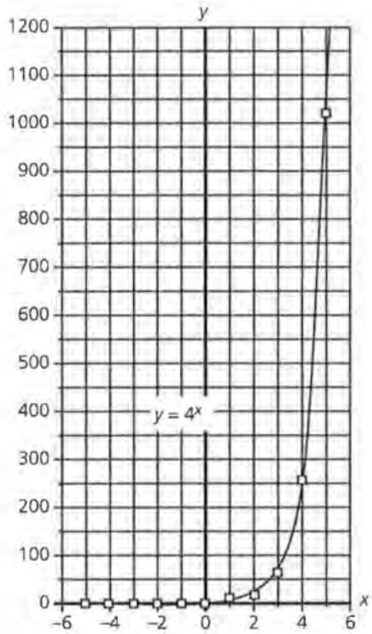

7. Sketch a graph of each function. You may use a graphing utility.

a. $y = 2^x$ b. $y = 3^x$

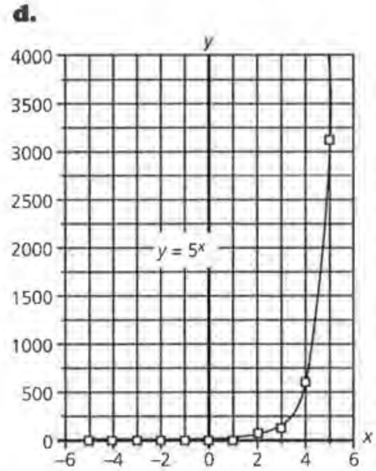
c. $y = 4^x$ d. $y = 5^x$

## STUDENT PAGE 25

**c.**



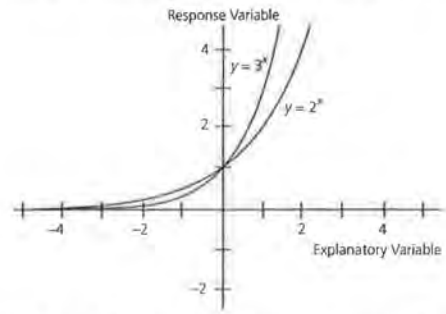

**d.**



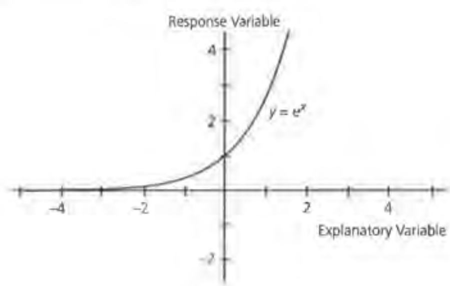

**8.** For $b > 1$, if the base $b$ of an exponential function is increased, the curve rises to the right more rapidly. For $0 < b < 1$, if the base $b$ increases, the curve rises to the left more slowly.

**8.** Describe how the graph of an exponential function changes as the base $b$ changes.

Two exponential functions are graphed simultaneously on the graph below to display the likeness in shape and the difference in the rate of increase. Although both functions increase to infinity as the explanatory variable increases, you will notice that $y = 2^x$ increases at a slower rate than $y = 3^x$ does.

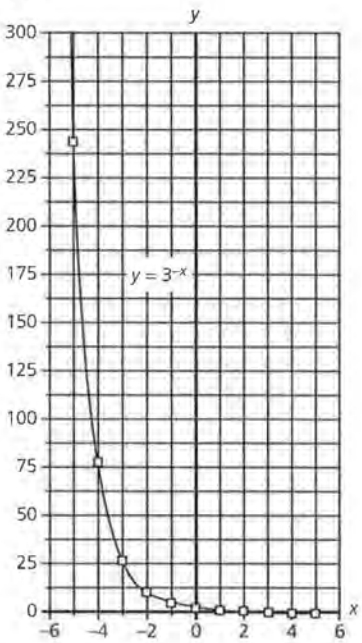



The symbol $e$ represents the numerical constant 2.7182818... . It is an irrational number defined as the limit of $(1 + \frac{1}{x})^x$ as $x$ gets larger and larger and approaches infinity. The letter $e$ was chosen for its discoverer, Leonard Euler. The graph of $y = e^x$ is shown below. You can see that its graph lies between the graphs of $y = 2^x$ and $y = 3^x$, since the value of $e$ lies between 2 and 3.

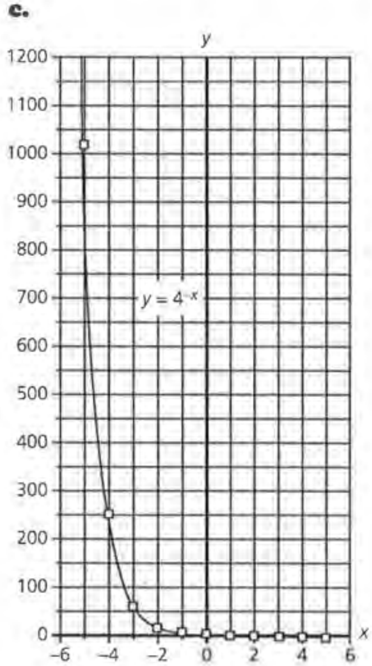


**9. a.** See graph below.

**b.**




**c.**




## STUDENT PAGE 26

**9.** Sketch a graph of each function. You may use a graphing utility.

**a.** $y = 2^{-x}$

**b.** $y = 3^{-x}$

**c.** $y = 4^{-x}$

**10.** What is the relationship between the graphs of the functions $y = 2^{-x}$ and $y = (\frac{1}{2})^{-x} = 2^x$?

In mathematical modeling, you must distinguish between a power function and an exponential function.

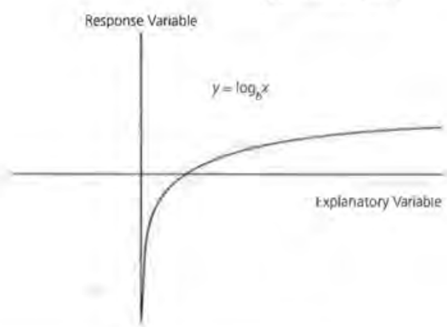
**11.** Graph the power function $y = x^2$ and the exponential function $y = 2^x$ on the same set of axes.

**a.** Describe in a short paragraph the differences between the graphs of these two functions.

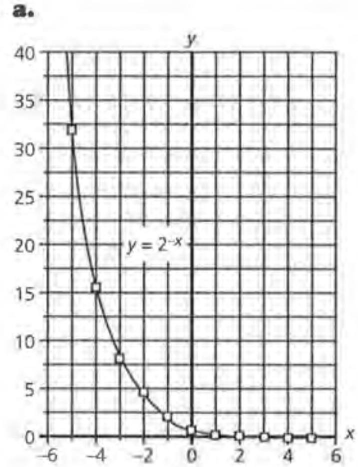
**b.** Describe a situation that could be modeled by $y = x^2$ and a situation that could be modeled by $y = 2^x$.

**12.** Without graphing, describe differences between $y = x^3$ and $y = 3^x$.

***Logarithmic Function*** The general form of the equation of a logarithmic function is $y = \log_b x$, where $b > 0$ and $b \neq 1$. The graph appears as a smooth curve and has a vertical asymptote as the value of the explanatory variable approaches zero.

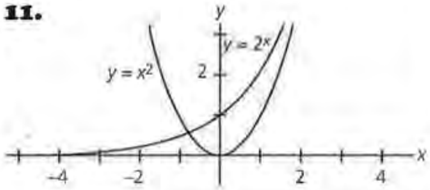



**a.**




**10.** They are reflections across the $y$-axis. The bases of the exponential function are multiplicative inverses.

**11.**



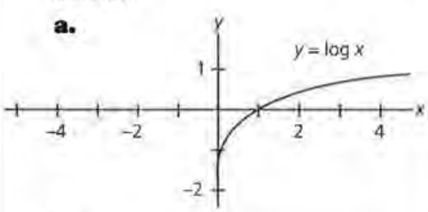

**a.** The function $y = 2^x$ is increasing over its entire domain and has the $x$-axis as a horizontal asymptote. The function $y = x^2$ is a decreasing function over the first half of its domain and is an increasing function over the second half. The function $y = x^2$ has no asymptotes.

**b.** Answers will vary. Students might describe some area problem to be modeled by the quadratic function and some growth or decay problem to be modeled by the exponential function. The model could be appropriate only for a specific domain.

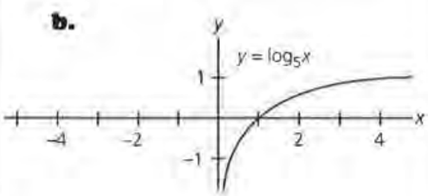
**12.** Both curves are increasing over their entire domain. The graph of $y = 3^x$ has the $x$-axis as an asymptote and never has negative response values. The graph of $y = x^3$ has negative response values whenever the explanatory variable is negative. The function $y = x^3$ has no asymptotes.

**13.** The rule for converting a logarithmic function from one base to another allows you to graph with different bases on a graphing calculator.

**a.**




**b.**

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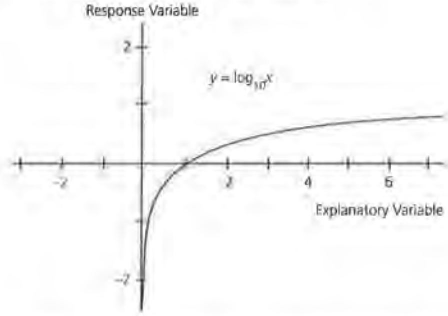
The general rule for converting a logarithmic function from one base to another is:

$\log_b x = \frac{\log_c x}{\log_c b}$

For example, $\log_2 6 = \frac{\log_{10} 6}{\log_{10} 2}$

When no base is indicated, such as in $\log x$, the base is understood to be 10; that is, $\log x = \log_{10} x$.




Another logarithmic function is so important in theoretical mathematics that it has its own symbol. It is the logarithm base $e$, written symbolically as $\ln x$: $\ln x = \log_e x$.

**13.** Use a graphing utility to graph each function.

**a.** $y = \log x$

**b.** $y = \log_5 x$

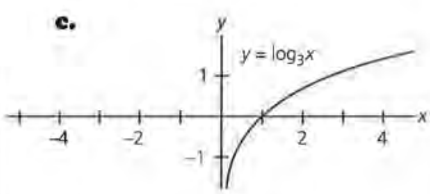
**c.** $y = \log_3 x$

**d.** $y = \log_{0.5} x$

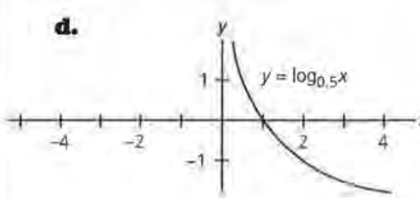
**e.** $y = \log_{0.75} x$

**f.** $y = \ln x$

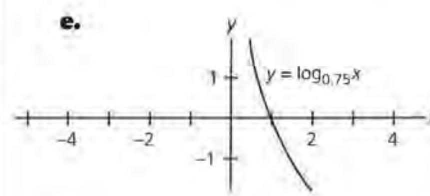
**c.**



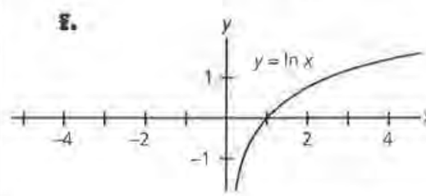

**d.**




**e.**

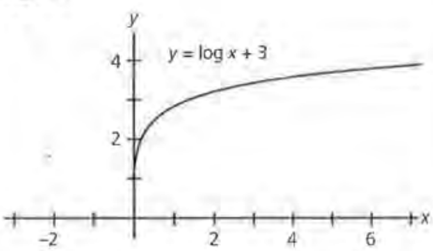



**f.**

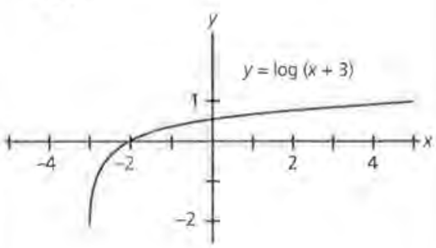


**14.** Every logarithmic function base $b$ goes through the points $(b, 1)$ and $(1/b, -1)$. So, for $b > 1$, increasing the value of $b$ produces a curve whose right end is closer to, yet still above, the $x$-axis. For $0 < b < 1$, increasing the value of $b$ produces a curve whose right end is farther away from, yet still below, the $x$-axis.

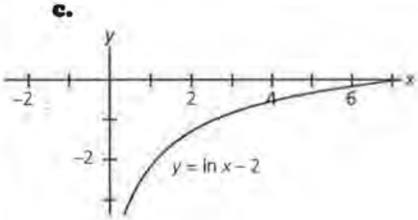
**15. a.**



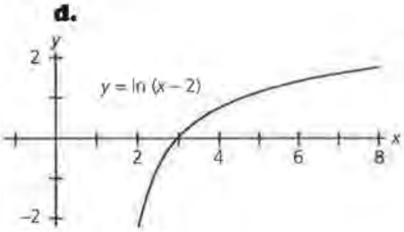

**b.**




**c.**




**d.**

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**14.** Describe the effect that changing the base has on the graph of a logarithmic function.

**15.** Use your graphing utility to graph each function.

**a.** $y = \log x + 3$

**b.** $y = \log(x + 3)$

**c.** $y = \ln x - 2$

**d.** $y = \ln(x - 2)$

**e.** $y = \log(x - 4) - 2$

**f.** $y = 2\log x$

**g.** $y = \log 2x$

**h.** $y = \log 2(x - 2)$

**16.** Explain how each constant $a$, $b$, $c$, and $d$ causes the graph of $y = a \log b(x - c) + d$ to differ from the graph of $y = \log x$.

**Summary**

The knowledge that the equation of a function and its graph are different representations of the same data set is very helpful in the process of modeling. A further knowledge of the effect the various constants have upon the graph of the function is helpful. In this unit, you investigated those items with respect to linear, power, exponential, and logarithmic functions. In the remainder of this module, you will use this knowledge to determine what function might be the best model for the data with which you are working.

**Practice and Applications**

For each of the following equations make a sketch of its graph. This is a mental exercise, and the graphing utility should be used only to check your results and relative accuracy.

**17.** $y = 3e^x + 1$

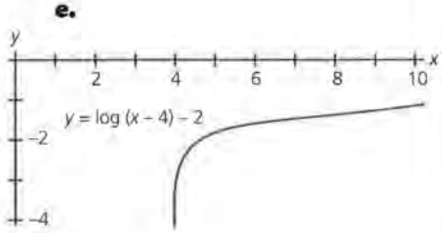
**18.** $y = 2 \log (x - 3) + 2$

**19.** $y = 3x + -2$

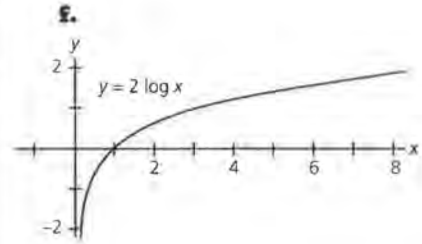
**20.** $y = 5(2)^{x+6} - 4$

**21.** $y = 2(3)^{2-x} + 1$

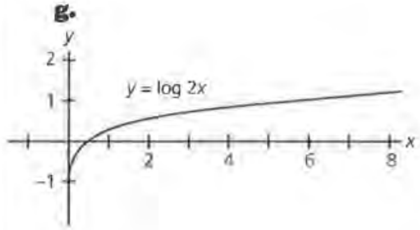
**e.**



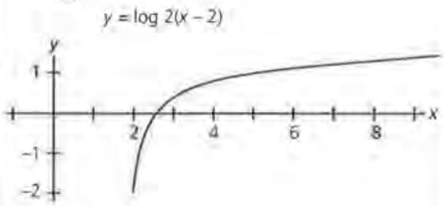

**f.**



**g.**



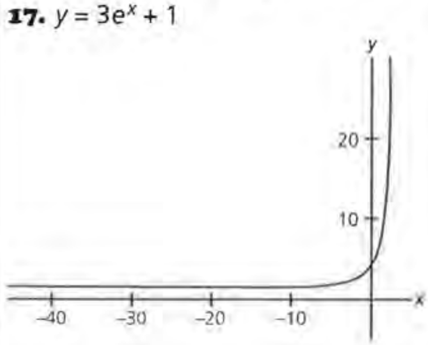

**h.**



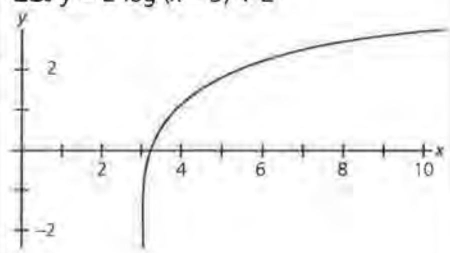

**16.** The constant $a$ causes a vertical stretching if $|a| > 1$ and a vertical shrinking if $0 < |a| < 1$; it does not change the $x$-intercept. If $a$ is negative, the curve is reflected across the $x$-axis. The constant $b$ has the corresponding horizontal effect. The effect of the constant $c$ is a horizontal translation of negative $c$ units, that is, $c$ units right when $c$ is negative and $c$ units left when $c$ is positive. The effect of the constant $d$ is to translate the curve vertically $d$ units.

## Practice and Applications

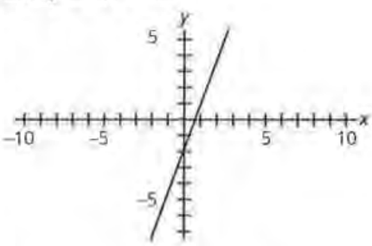
**17.** $y = 3e^x + 1$



**18.** $y = 2 \log (x - 3) + 2$

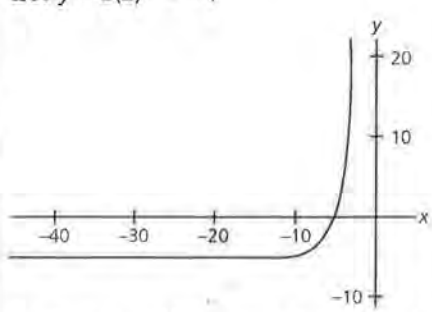


**19.** $y = 3x + -2$



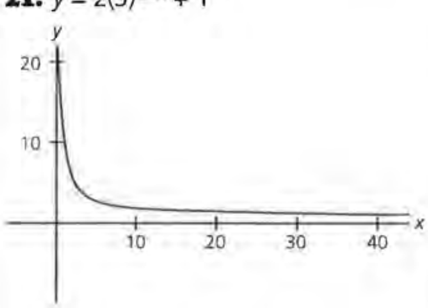
For Question 20, suggest that students apply these tranformations to $y = 2^x$: Stretch by a factor of 5, shift 6 units left, shift 4 units down.

**20.** $y = 5(2)^{x+6} - 4$



For Question 21, suggest that students apply these tranformations to $y = 3^{-x}$: Stretch by a factor of 2, shift 2 units left, shift 1 unit up.

**21.** $y = 2(3)^{2-x} + 1$



## LESSON 4

# Patterns in Graphs

**Materials:** graph paper, rulers
**Technology:** graphing calculators or computer
**Pacing:** 1 class period

### Overview

In this lesson, students will practice identifying linear, nonlinear, increasing, and decreasing patterns in graphs. A mathematical model is an equation used to describe the response variable in terms of the explanatory variable. If the data set to be modeled has the specific characteristic that a straight line would "best" describe the pattern formed by its points, it calls for a ***linear model.*** If the pattern seems more curved, the data set would be described with a ***nonlinear model.*** If the data set's response variable increases in value while the explanatory variable increases in value, the model being called for is an ***increasing model.*** If the response variable decreases in value while the explanatory variable increases in value, a ***decreasing model*** is appropriate.

### Teaching Notes

It is important to allow students to identify the properties of graphs from the graph, the table of values, and the symbolic representation. Although this lesson will not take a great deal of time, students must be allowed time to develop these skills if they have not already done so. If they have already developed these skills, this lesson could be omitted.

### Follow-Up

Have students conjecture which of the functions reviewed in Lesson 3 would be possible models of each of these data sets. Be sure that they recognize that in many cases there will be more than one possibility. This will support the need to determine which might be considered the best fit, a concept that will be studied in subsequent lessons.

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LESSON 4

# Patterns in Graphs

**Why are mathematical models used to describe data?**

**Are there any common patterns that appear in graphs of functions?**

A mathematical model is an equation used to describe the response variable in terms of the explanatory variable. If the data set to be modeled has the specific characteristic that a straight line would best describe the pattern formed by its points, it calls for a ***linear model.*** If the pattern seemed more curved, the data set would be said to be calling for a ***nonlinear model.*** If the data set's response variable increases in value while the explanatory variable increases in value, the model being called for is an ***increasing model.*** If the response variable decreases in value while the explanatory variable increases in value, the model called for is a ***decreasing model.***

**OBJECTIVES**

Define a mathematical model and explore different data sets.

Identify which data sets can be represented by linear and nonlinear models.

Make suggestions regarding probable models.

**INVESTIGATE**

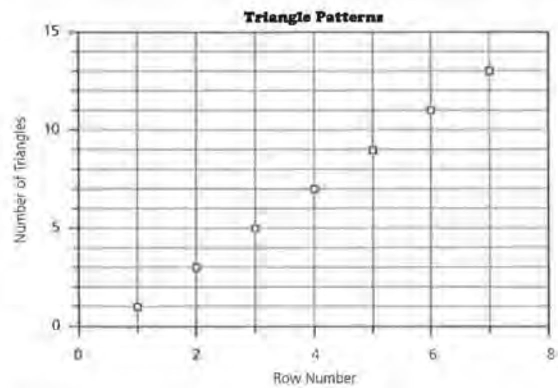
When you investigated the patterns in the triangle in Lesson 1, you considered (row number, number of triangles). The results looked like the graph on the following page.

## Solution Key

### Discussion and Practice

1. If the data set represents ordered pairs in which the response variable (domain) values increase while the explanatory variable (range) values increase, the model is increasing regardless of its being linear or nonlinear.
2. If the data represent ordered pairs in which the response variable values increase while the explanatory variable values decrease, it will be a decreasing model independent of its being linear.

## STUDENT PAGE 30




In this case, the graph appears to represent a data set that would call for a linear model. Is it obvious that it also calls for an increasing model?

When reading information in newspapers and magazines or watching the news on TV, you will often encounter information in the form of a table. This information may be used to answer a question, describe a trend, or tell a story.

**Discussion and Practice**

1. Is it possible for a data set to be represented by a nonlinear model and also be an increasing model? Explain.
2. Is it possible for a data set to be represented by a linear model and also be a decreasing model? Explain.

**Summary**

Recognition of linear and nonlinear models as well as increasing and decreasing models is part of the mathematical modeling process.

**Practice and Applications**

The following tables and graphs provide information that may follow a pattern. For each table or graph in Questions 3–14, look for patterns following this procedure:

a. Create a scatter plot for any data set that does not already have a graph.

STUDENT PAGE 31

## Practice and Applications

**3.** This scatter plot appears to be increasing and straight. It would require a linear model.

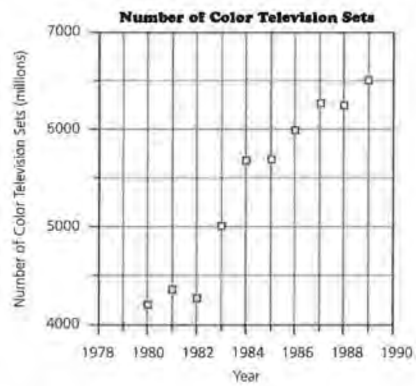
**4.** This scatter plot appears to be increasing and nonlinear. It could possibly be modeled with a quadratic, cubic, or exponential model.

**b.** Examine each scatter plot and identify linear and nonlinear models. You may use a graphing utility to make your scatter plots.

**c.** In your examination, determine which of the model functions are classified as increasing and which are classified as decreasing.

**d.** After you have described the characteristics, use the information you gained in Lesson 3 to suggest one or two types of functions that could be possible models for that data set.

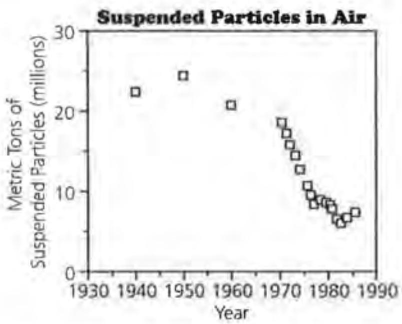
**3.**




**4.**

5. This scatter plot appears to be an increasing and nonlinear model. It could possibly be modeled with a quadratic or logarithmic model.

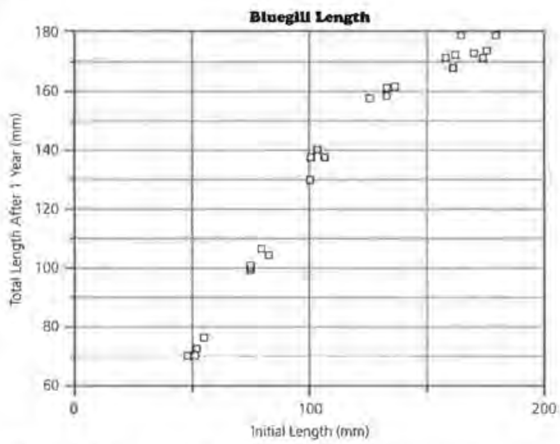
6.




This scatter plot appears to be increasing over certain portions of the domain and decreasing over other portions. It could be modeled with a nonlinear model, possibly a cubic.

## STUDENT PAGE 32

5.



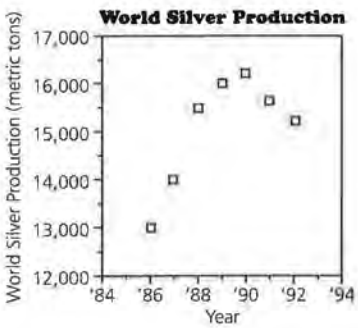

6. The data below show the amount of suspended particulate matter in millions of metric tons emitted into the air for various years in the United States.

| Year | Metric Tons of Suspended Particles in the Air (millions) |
|---|---|
| 1940 | 22.8 |
| 1950 | 24.5 |
| 1960 | 21.1 |
| 1970 | 18.1 |
| 1971 | 16.7 |
| 1972 | 15.2 |
| 1973 | 14.1 |
| 1974 | 12.4 |
| 1975 | 10.4 |
| 1976 | 9.7 |
| 1977 | 9.1 |
| 1978 | 9.2 |
| 1979 | 9.0 |
| 1980 | 8.5 |
| 1981 | 7.9 |
| 1982 | 7.0 |
| 1983 | 6.7 |
| 1984 | 7.0 |
| 1985 | 7.3 |

Source: *USA by Numbers, Zero Population Growth, Inc.*, 1988

## STUDENT PAGE 33

**7.**




This scatter plot is increasing on part of its domain, decreasing over another, and nonlinear. It could possibly be modeled with a quadratic model.

**8.** This scatter plot is increasing and nearly straight. It could be modeled with a linear model.

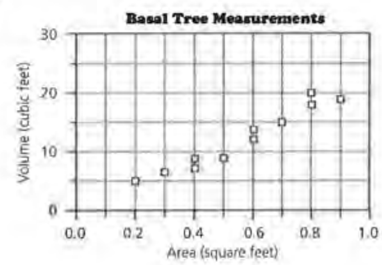
**9.** This scatter plot is increasing and nonlinear. It could possibly be modeled with a quadratic, cubic, or exponential model.

**7.**

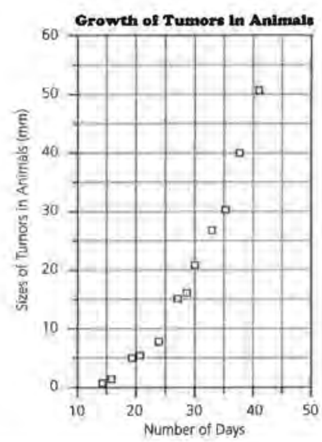
| Year | World Silver Production (metric tons) |
|---|---|
| 1986 | 12,970 |
| 1987 | 14,019 |
| 1988 | 15,484 |
| 1989 | 16,041 |
| 1990 | 16,216 |
| 1991 | 15,692 |
| 1992 | 15,345 |

Source: *World Almanac*, 1995

**8.**




**9.**



## STUDENT PAGE 34

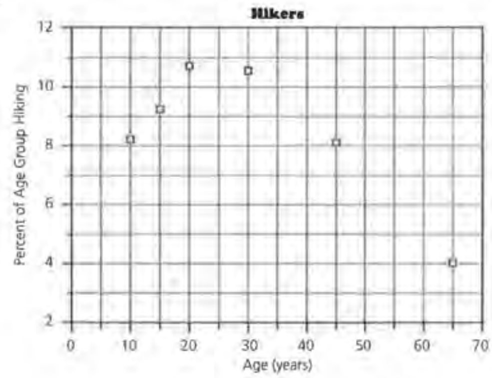
**10.** This scatter plot is increasing over part of the domain and decreasing over another. It requires a nonlinear model. It could possibly be modeled with a quadratic or cubic model.

**11.** This scatter plot is increasing over part of the domain, decreasing over another, and nonlinear. It could possibly be modeled with a quadratic or cubic model.

**12.** This scatter plot is increasing and nearly straight. It could possibly be modeled with a linear model.

**10.**

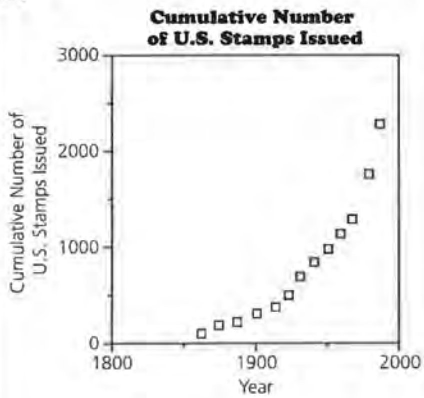

**11.**




**12.**

**13.** This scatter plot is increasing and nonlinear. It could possibly be modeled with a quadratic, cubic, or exponential model.

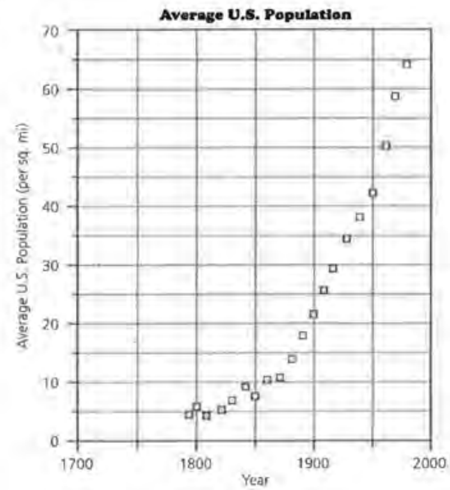
**14.**




This scatter plot is increasing and nonlinear. It could possibly be modeled with a quadratic, cubic, or exponential model.

## STUDENT PAGE 35

**13.**




**14.**

| Year | Cumulative Numbers of U.S. Stamps Issued |
|---|---|
| 1868 | 88 |
| 1878 | 181 |
| 1888 | 218 |
| 1898 | 293 |
| 1908 | 341 |
| 1918 | 529 |
| 1928 | 647 |
| 1938 | 838 |
| 1948 | 980 |
| 1958 | 1123 |
| 1968 | 1364 |
| 1978 | 1769 |
| 1988 | 2400 |

Source: *Scotts Standard Postage Stamp Catalog*, 1989

LESSON 5

# Transforming Data

**Materials:** graph paper, rulers
**Technology:** graphing calculators or computer (optional)
**Pacing:** 3 class periods

## Overview

In this lesson, students will investigate the transformation of the units on one of the axes. Transformation will be defined and discussed. Students will investigate what will happen to a graph's appearance when a transformation is applied to one of the units.

## Teaching Notes

One of the most powerful and useful approaches in mathematical modeling is to *linearize* a graph or actually linearize a data set by applying a transformation to one of the units prior to plotting. The results of a transformation are: (1) The distances between the data points are changed. For example, in the (circumference squared, area) graph, the horizontal distances between the data points increase as $x$ increases. (2) A transformation alters the appearance of the graph of the data, often changing its shape. The students should be led to discover the relationships. Resist the temptation to tell them the results prior to their discovery.

STUDENT PAGE 36

LESSON 5

# Transforming Data

**What are some things that will affect the shape of a graph?**

**What effect would transforming data have on the graph of the data?**

As you know, *pi* ($\pi$) is the ratio of the circumference of a circle to its diameter, or 3.14159.... It is used in formulas that describe circular figures.

OBJECTIVE

Transform specific data sets to make them appear linear when plotted in a scatter plot.

INVESTIGATE

In the Old Testament of the *Bible* (II Chronicles 4:2), it is stated, "Then he made the molten sea; it was round, ten cubits from brim to brim, and five cubits high, and a line of thirty cubits measured its circumference." The circumference was, therefore, 6 times the radius or 3 times the diameter. The Hebrews used 3 for $\pi$. The Egyptians used $\sqrt{10}$ or 3.16. Which value of $\pi$ do you commonly use?

**Discussion and Practice**

1. Use $\pi = 3.14159$ to complete the table on page 37 about circles.

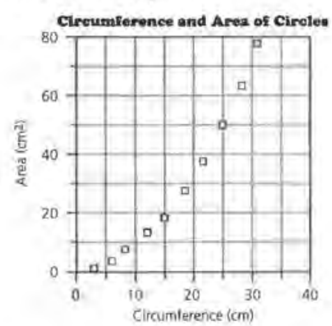
STUDENT PAGE 37

| Diameter (cm) | Area ($cm^2$) | Circumference (cm) |
|---|---|---|
| 1 | ____ | ____ |
| 2 | ____ | ____ |
| 3 | ____ | ____ |
| 4 | ____ | ____ |
| 5 | ____ | ____ |
| 6 | ____ | ____ |
| 7 | ____ | ____ |
| 8 | ____ | ____ |
| 9 | ____ | ____ |
| 10 | ____ | ____ |

**a.** Make a scatter plot of (diameter, area).

**b.** Make a scatter plot of (diameter, circumference).

**2.** The data from the table above were used to make the (circumference, area) scatter plot below. Compare the (circumference, area) scatter plot to your (diameter, area) and (diameter, circumference) scatter plots. How are they alike or different? Explain.




In previous lessons, we analyzed the changes that occurred in graphs when the unit measure of either or both of the variables was changed.

**3.** Explain in a short paragraph what effect a unit change can have on a graph.

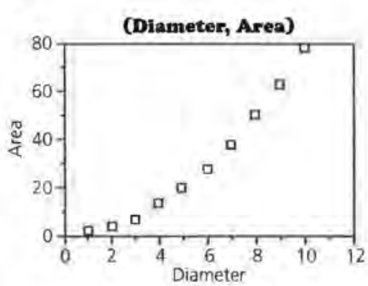
## Solution Key

## Discussion and Practice

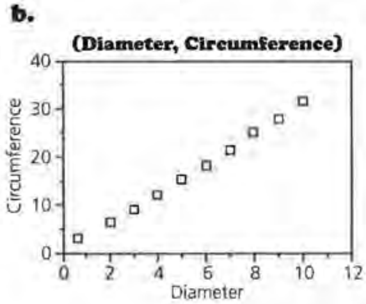
**1.** Answers will vary. The most likely answer is 3.14.

| Diameter (cm) | Area ($cm^2$) | Circumference (cm) |
|---|---|---|
| 1 | 0.785 | 3.142 |
| 2 | 3.142 | 6.283 |
| 3 | 7.069 | 9.425 |
| 4 | 12.566 | 12.566 |
| 5 | 19.635 | 15.708 |
| 6 | 28.274 | 18.850 |
| 7 | 38.485 | 21.991 |
| 8 | 50.265 | 25.133 |
| 9 | 63.617 | 28.274 |
| 10 | 78.540 | 31.416 |

**a.**




**b.**




**2.** The graph (diameter, area) and the graph (circumference, area) are both curved and concave upward. The horizontal scale in the two graphs is different, resulting in an appearance that the (diameter, area) graph is increasing more rapidly. This is described mathematically as horizontal stretch or shrink. The change from diameter to circumference is a scale change and can be compared to changing from dimes to quarters on the explanatory axis. This was observed in Lesson 2.

**3.** Changing the units will cause a distortion of the graph. It can make the graph appear steeper or shallower and imply a different rate of change than the actual.

4. Answers will be an original graph. The purpose is to have students experience the difficulty in creating the scale in such a graph to better appreciate how the process of transformation of scale enables easier graphing.

5. Answers will vary. Sample: Spread the graph to enable reading or interpolation.

## STUDENT PAGE 38

In this lesson, you will analyze the changes on a graph produced by a *transformation* performed on a variable. A transformation on a variable is accomplished by using a function to change its value. You are familiar with the effect of unit changes on a graph. For example, changing from quarters to dollars or changing from people to thousands of people are examples of unit changes. Such changes can expand or contract a graph, but the distances between tick marks on the scale remain uniform.

Some situations force us to consider other kinds of scale changes. For example, consider the data of the intensity of sound (number of times as loud as the softest sound) in decibels.

| Sample Sound | Decibels | Number of Times as Loud as Softest Sound |
|---|---|---|
| Jet airplane | 140 decibels | 100,000,000,000,000 |
| Air raid siren | 130 decibels | 10,000,000,000,000 |
| Pneumatic hammer | 120 decibels | 1,000,000,000,000 |
| Bass drum | 110 decibels | 100,000,000,000 |
| Thunderclap | 100 decibels | 10,000,000,000 |
| Niagara Falls | 90 decibels | 1,000,000,000 |
| Loud radio | 80 decibels | 100,000,000 |
| Busy street | 70 decibels | 10,000,000 |
| Hotel lobby | 60 decibels | 1,000,000 |
| Quiet automobile | 50 decibels | 100,000 |
| Average residence | 40 decibels | 10,000 |
| Average whisper | 30 decibels | 1,000 |
| Faint whisper | 20 decibels | 100 |
| Rustling leaves or paper | 10 decibels | 10 |
| Softest sound heard | 0 decibels | 1 |

Source: *Definitions of Integrated Circuits, Logic, and Microelectronics Terms*

4. Use the data above to make a graph of (number of times as loud as softest sound, decibels). Determine your own scale and label the axes.

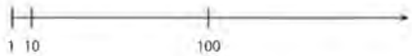
5. What problems did you have in choosing the scale for the graph in Question 4?

What suggestions do you have for changing the scale to make the graph easier to read?

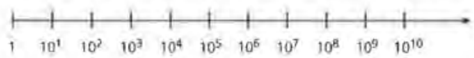
To put all of the points on the graph, the graph had to range from 0 to 100,000,000,000,000 on the horizontal axis. This is impractical because of the relative size of the largest and smallest intervals in the $x$-range. On the first number line on page 39,

# STUDENT PAGE 39

there are 9 units between 1 and 10, and 90 units between 10 and 100. The distance between the $x$-values 1 and 10, 10 and 100, 100 and 1000, and so on, increases as $x$ increases.

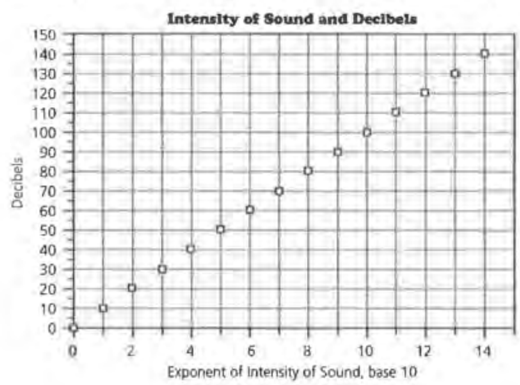



It is reasonable to consider a change that would compress the largest intervals in the $x$-range without also compressing the smallest intervals. One way to preserve the values of the points plotted and create a scale in which the horizontal values are the same distance apart is to use a scale containing the numbers represented in exponential form. Consider the following scale:




This type of scale change, in which the distances between points appear equal but actually represent different values, is an example of a transformation.

When the $x$-axis is transformed, the graph of (exponent of intensity of sound, decibels) changes as shown in this graph.




To enable us to graph the (exponent of intensity of sound, decibels) ordered pairs and compensate for the rapid increase in horizontal values, each intensity-of-sound value has been mathematically transformed.

6. The use of this transformation resulted in a graph that is linear. The horizontal scale makes the plotting of each point a unique point, distinguishable from one another.

7. $y = 10x$; that is, decibels = 10 × (exponent, base 10, of the intensity).

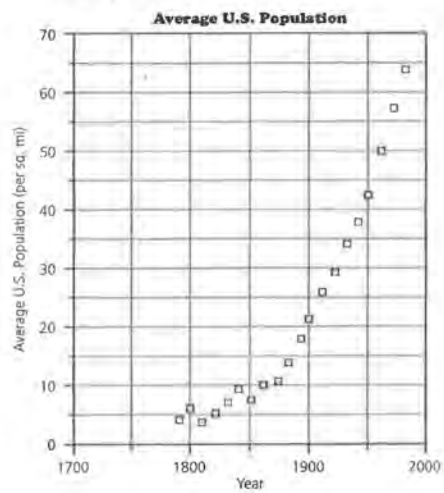
## STUDENT PAGE 40

6. Write a paragraph describing the changes you observe.

7. Write an equation to describe decibels in terms of the exponent, base 10, of the intensity of sound.

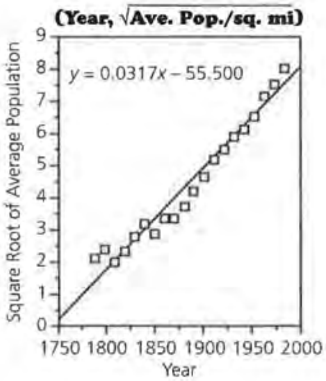
Straightening a graph through transformation is often an advantage to us, because it is much simpler to create the equation of a straight line than that of a curve. When we are able to ***linearize*** a data set by performing a transformation on the units, we can fit a straight line and create the equation of that line. The line or its equation can be used to describe patterns and make predictions about the data set.

Here is an example of how the transformation of units can linearize a graph.



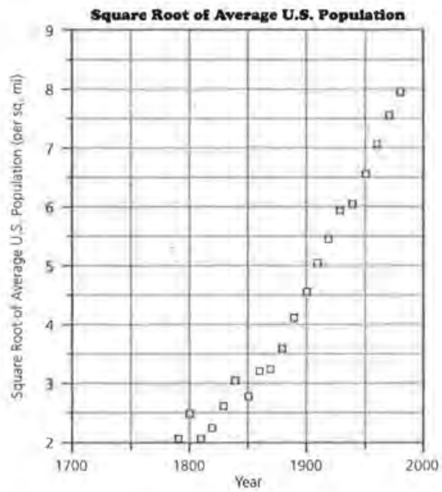

Transform the data on the vertical axis above by taking the square root of each unit. The results are plotted using the new values on the vertical axis in the graph on page 41.

8.




9. Answers will vary. Example: $y = 0.0317x - 55.5$

# STUDENT PAGE 41




You will notice a straightening effect, or linearization, of the curve created by the transformation.

8. On the grid on *Activity Sheet 1*, draw a straight line that seems to be a good fit for the previous graph that models the relationship between the square root of the average population per square mile and the year.

9. Write an equation for the line of best fit for the square root of average population per square mile in terms of year.

**Summary**

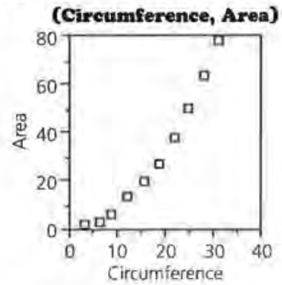
In this lesson, you investigated the transformation of the units on one of the axes. A transformation of one axis is accomplished by applying a function to the $x$- or $y$-values. The results of a transformation are:

(1) The distances between the data points are changed. For example, in the (exponent of intensity of sound, decibels) graph in this lesson, the horizontal distances between the data points increase as $x$ increases.

STUDENT PAGE 42

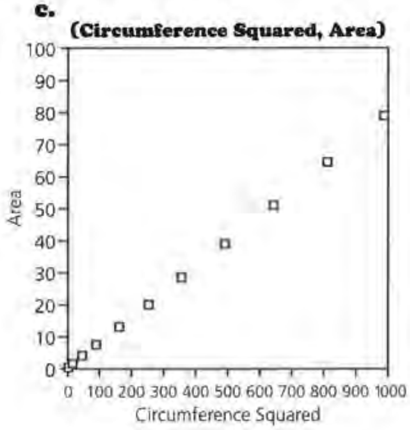
## Practice and Applications

**10. a.**




**b.** See table below.

**c.**




(2) A transformation alters the appearance of the graph of the data, often changing its shape.

We will now investigate specific methods of effecting changes in graphs.

**Practice and Applications**

**10.** Use the data table at the beginning of this lesson.

**a.** Make a scatter plot of (circumference, area).

**b.** Add a new column to the data table for circumference squared.

**c.** Make a scatter plot of (circumference squared, area).

**d.** Describe how the graph in part c is different from the graph in part a.

**11.** Write an equation for area in terms of the circumference squared.

| Diameter (cm) | Area ($cm^2$) | Circumference (cm) | Circumference Squared ($cm^2$) |
|---|---|---|---|
| 1 | 0.785 | 3.142 | 9.870 |
| 2 | 3.142 | 6.283 | 39.478 |
| 3 | 7.069 | 9.425 | 88.826 |
| 4 | 12.566 | 12.566 | 157.914 |
| 5 | 19.635 | 15.708 | 246.740 |
| 6 | 28.274 | 18.850 | 355.306 |
| 7 | 38.485 | 21.991 | 483.611 |
| 8 | 50.265 | 25.133 | 631.654 |
| 9 | 63.617 | 28.274 | 799.438 |
| 10 | 78.540 | 31.416 | 986.960 |

**d.** The second graph appears linear in comparison to the first. Notice that the horizontal spacing between data points has changed. The (circumference, area) graph was stretched horizontally to create the (circumference squared, area) graph.

**11.** Area = $\frac{c^2}{4\pi}$

## LESSON 6

# Exploring Changes on Graphs

**Materials:** graph paper, rulers, *Activity Sheets 1–3,* Unit II Test
**Technology:** graphing calculators or computer
**Pacing:** 3 class periods

### Overview

The association between a graph's shape and the scale on either axis is another important relationship in the process of modeling. In this lesson, students will investigate that relationship and become aware of the choice of the inverse of a function to effect the straightening of the curve.

### Teaching Notes

In this lesson, students will bring together the ideas of unit and scale change and the concepts and properties of functions, function inverses, and their graphs and behaviors. They should become aware of the power of transformation in the art of mathematical modeling. Transforming one of the units of a scatter plot using the inverse function is a very powerful and useful tool. Students should discover that such a transformation often linearizes the graph and allows for the application of the knowledge of linear equations to understand the scatter plot more effectively. It is advantageous to linearize a data set because it is easier to make conclusions with a linear model.

### Follow-Up

Students can be provided with, or asked to research, data sets whose scatter plots are nonlinear. They should then proceed to discover what transformation on that data would linearize its graph. They could be asked to conjecture which transformation they think might work and why, and then actually perform that transformation and either confirm or deny the validity of their conjecture.

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LESSON 6

# Exploring Changes on Graphs

**What effect does changing the scale on an axis have on the graph of the data?**

**Is it true that a transformation using a function's inverse will linearize that function's graph?**

## INVESTIGATE

Soccer balls come in different sizes, which are indicated by numbers.

How do volume and circumference compare?

| Size | Circumference (cm) | Volume ($cm^3$) | Age of Player |
|---|---|---|---|
| 3 | 59 | 3468.2 | under 8 |
| 4 | 62.8 | 4182.4 | 8–11 years |
| 5 | 67 | 5078.9 | 12 years and older |

Other sports also have balls with standardized circumferences.

| Ball | Circumference (cm) | Volume ($cm^3$) |
|---|---|---|
| Soccer, size 3 | 59 | 3468.2 |
| Soccer, size 4 | 62.8 | 4182.4 |
| Soccer, size 5 | 67 | 5078.9 |
| Softball | 33 | 606.9 |
| Basketball | 82.5 | 9482.2 |
| Golf Ball | 13.5 | 41.5 |
| Playground Ball | 69 | 5547.5 |
| Racquetball | 18 | 98.5 |
| Tennis Ball | 20.2 | 139.2 |
| Baseball | 23.3 | 213.6 |

**OBJECTIVE**

Recognize and understand how the shape of a graph changes when a variable plotted in the graph is transformed.

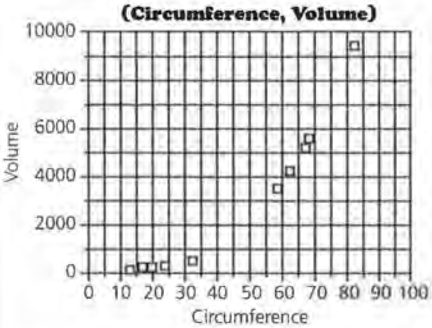
## Solution Key

## Discussion and Practice

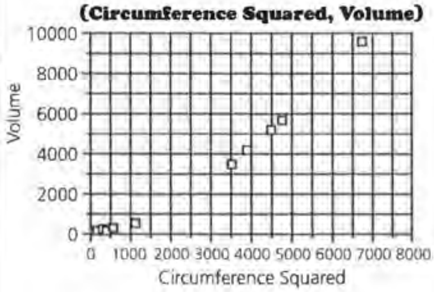
**1.** Students may use a graphing calculator as an aid, but should be required to plot the graphs on paper so that they recognize the subtle differences among them.

The comparison is that the first three graphs appear nonlinear, while the fourth and fifth plot appear to be linear. The transformation of squaring applied to the circumference resulted in a lessening of the curvature but not a complete straightening. The transformation of square root applied to the volume had a similar effect with a greater lessening of curvature than the first but still not linear. The transformation of cubing the circumference had the effect of straightening the curve. The transformation of cube root applied to the volume also had the effect of linearizing the plot.

**(Circumference, Volume)**




**(Circumference Squared, Volume)**




# STUDENT PAGE 44

**Discussion and Practice**

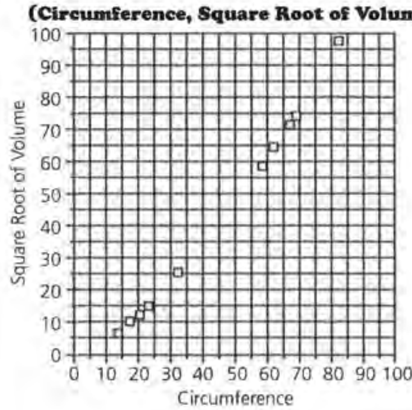
Use the data on page 43 to make five scatter plots: (circumference, volume), (circumference squared, volume), (circumference, square root of volume), (circumference cubed, volume), and (circumference, cube root of volume) on the grids provided on *Activity Sheets 2–4*. Use the graphs to answer the following questions.

1. Compare and then explain how transformations change the appearance of the graph.
2. Which graph(s) are easier to describe algebraically? Explain.
3. Compare the first graph to the fourth graph. How did the transformation of circumference into circumference cubed change that graph? How did the transformation of volume into cube root of volume in the last graph change the first graph?
4. Make a conjecture about your findings in Question 3.
5. Why do you think cube and cube-root transformations might be preferred over the square and square-root transformations in this example?
6. Classify each of the above graphs as linear or nonlinear and increasing or decreasing.
7. Measure the circumference of a beach ball in centimeters and use one of the graphs to predict the volume.
   a. Which graph did you select for this purpose? Why?
   b. Compute the volume of the beach ball and determine how close to the actual volume your prediction came.
8. Write an equation to represent each of following relations.
   a. The cube root of the volume of a sphere in terms of its circumference
   b. The volume of a sphere in terms of its circumference cubed

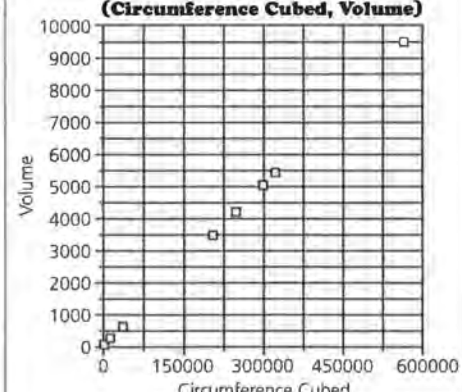
**Summary**

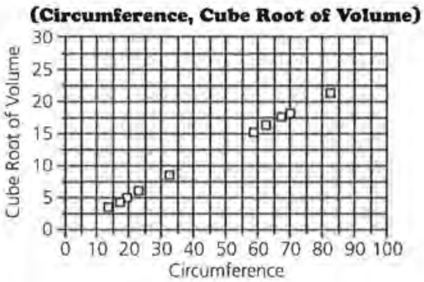
The association between a graph's shape and the scale on either axis is another important relationship in the process of modeling. In this unit, you investigated that relationship and became aware of the choice of the inverse of a function to effect the straightening of the curve.

**(Circumference, Square Root of Volume)**




**(Circumference Cubed, Volume)**






**2.** The two graphs (circumference$^3$, volume) and (circumference, cube root of volume) would be easiest to describe algebraically because they are linear.

**3.** From the first graph to the fourth graph, the transformation linearized the graph by stretching the horizontal axis or scale. The transformation of the volume into cube root of volume also linearized the graph, but this time it was accomplished by shrinking the vertical axis or scale.

**4.** Answers will vary. Sample: When there is an $n$th power relationship between a response variable and an explanatory variable, there will be a linear relationship between the response variable and the $n$th root of the explanatory variable.

**5.** These transformations are preferred because volume is measured in cubic units.

**6.** The first three graphs are nonlinear and increasing. The fourth and fifth graphs are linear and increasing.

**7.** **a.** Answers will vary.
**b.** Answers will vary.

**8.** Let $V$ be the volume of a sphere with circumference $C$.

**a.** $\sqrt[3]{V} = \sqrt[3]{\frac{1}{6\pi^2}}\, C$

**b.** $V = \frac{C^3}{6\pi^2}$

## Practice and Applications

**9. a.**

| Ancestors | Generations Ago |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |
| 8 | 256 |
| 9 | 512 |
| 10 | 1024 |

| Ancestors | Generations Ago |
|---|---|
| 1 | $2^1$ |
| 2 | $2^2$ |
| 3 | $2^3$ |
| 4 | $2^4$ |
| 5 | $2^5$ |
| 6 | $2^6$ |
| 7 | $2^7$ |
| 8 | $2^8$ |
| 9 | $2^9$ |
| 10 | $2^{10}$ |

**b.** The graphs both use the same unit scale on the horizontal axis, and the vertical axis expresses the same values in different units. The first graph appears to be nonlinear and the second appears to be linear.

# STUDENT PAGE 45

**Practice and Applications**

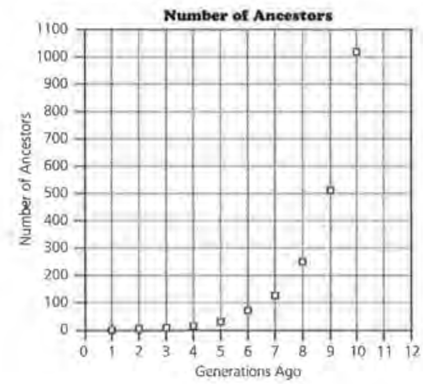
**Ancestors Problem**

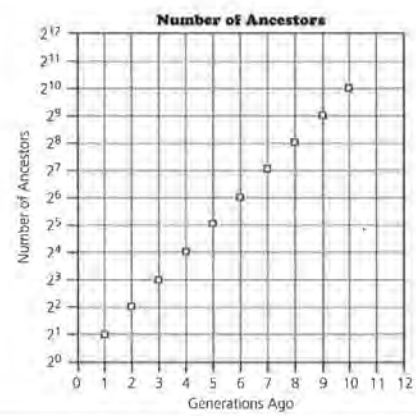
In the ancestors problem of Lesson 1, you made a scatter plot of (number of ancestors, generations ago).

**9.** Study the following two graphs of the ancestors problem.

**a.** Make a list of the ordered pairs graphed in each one.

**b.** Explain how the graphs are alike and how they are different.



**10.** $y = 2^x$

**11.** The $y$-values were transformed by expressing the response variable as a power of the base 2. The transformation linearizes the graph.

## STUDENT PAGE 46

**10.** Write an equation representing the number of ancestors in terms of the exponent of the generations ago.

Notice that although the physical distances between the vertical tick marks, $2^1$ and $2^2$, and $2^2$ and $2^3$, and so on, appear equal, the numerical distances between them are not equal.

**11.** Were the $x$-values or the $y$-values transformed to create the second graph? Describe the effect of this transformation on the graph.

The decibel data from Lesson 5, repeated here, has been used to create the two graphs on the next page.

| Sample Sound | Decibels | Number of Times as Loud as Softest Sound |
|---|---|---|
| Jet airplane | 140 decibels | 100,000,000,000,000 |
| Air raid siren | 130 decibels | 10,000,000,000,000 |
| Pneumatic hammer | 120 decibels | 1,000,000,000,000 |
| Bass drum | 110 decibels | 100,000,000,000 |
| Thunder clap | 100 decibels | 10,000,000,000 |
| Niagara Falls | 90 decibels | 1,000,000,000 |
| Loud radio | 80 decibels | 100,000,000 |
| Busy street | 70 decibels | 10,000,000 |
| Hotel lobby | 60 decibels | 1,000,000 |
| Quiet automobile | 50 decibels | 100,000 |
| Average residence | 40 decibels | 10,000 |
| Average whisper | 30 decibels | 1,000 |
| Faint whisper | 20 decibels | 100 |
| Rustling leaves or paper | 10 decibels | 10 |
| Softest sound heard | 0 decibels | 1 |

Source: *Definitions of Integrated Circuits, Logic, and Microelectronics Terms*

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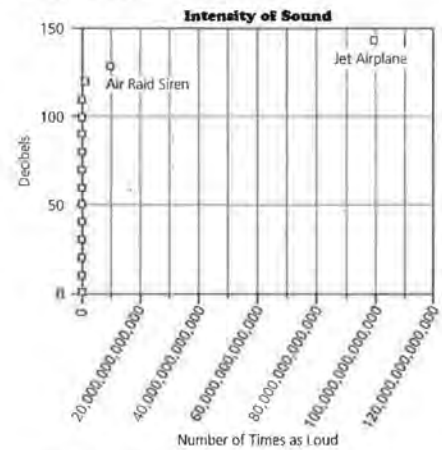
**12. a.** The numbers on the horizontal scale are one hundredth of the corresponding numbers on the scale of the first graph.

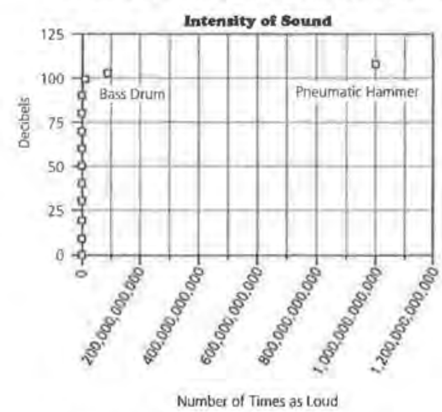
**b.** The plot appears the same except for the scales.

**c.** No

Study this graph of the decibel data.




**12.** The plot above looks linear except for two points. Consider the following plot with the data having a vertical scale from 0 to 125 decibels to eliminate the two highest data points.




**a.** How has the horizontal scale changed?

**b.** How does the appearance of the graph change?

**c.** Is the graph easier to read?

**13.** The graph would appear the same as the other two graphs.

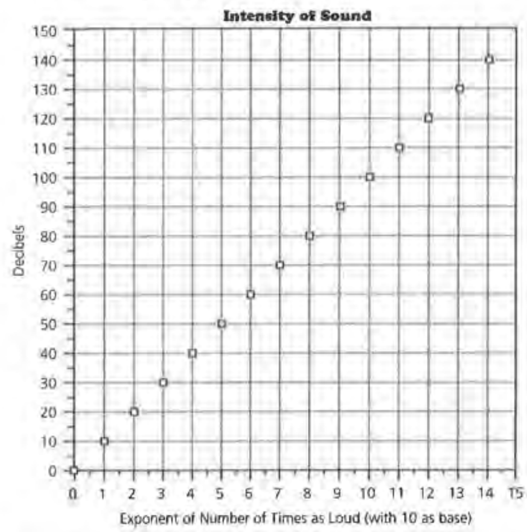
**14.** Yes; the graph does represent the data table. The appearance is different because the horizontal scale has been transformed to create an equal space between consecutive points when graphed.

**15.** According to the graph the exponent is 8, or $10^8 = 100{,}000{,}000$ times as loud.

## STUDENT PAGE 48

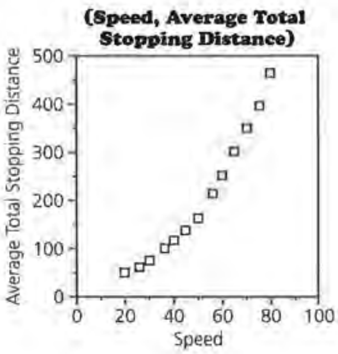
**13.** Describe what happens when you draw the graph with a vertical scale from 0 to 100 and eliminate two additional points.

**14.** Does the following graph accurately describe the data in the decibels table? Justify your answer.



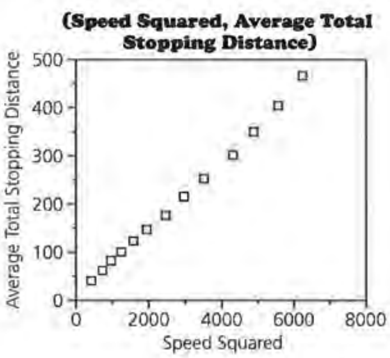

**15.** Use the graph above. For a decibel level of 80, what is the intensity of sound?

**1.**

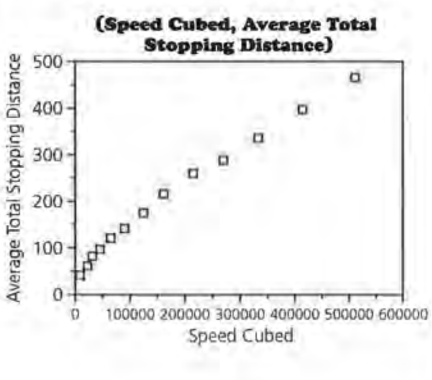



The graph is increasing and non-linear, probably quadratic.

**2.** **a.** $d = 0.069s^2 + 6.20$, where $d$ = distance and $s$ = speed




**b.** $d = 0.000826s^3 + 57.5$, where $d$ = distance and $s$ = speed

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ASSESSMENT

# Stopping Distances

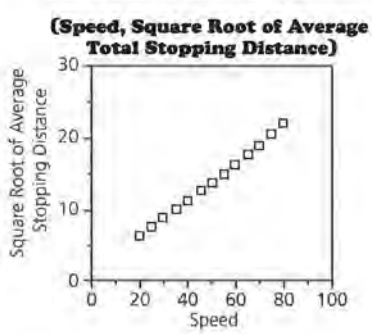
Driving students are usually taught to allow one car length, or about 15 feet, between their car and the next car for every ten miles of speed under normal driving conditions and a greater distance in adverse weather or road conditions. The faster a car is traveling, the longer it takes the driver to stop the car. The stopping distance depends on the driver-reaction distance and the braking distance. The total stopping distance is equal to the sum of the distance the car travels in the time it takes the driver to react and the distance the car travels after the brakes are applied.

| Speed (mph) | Average Total Stopping Distance (ft) |
|---|---|
| 20 | 42 |
| 25 | 56 |
| 30 | 73.5 |
| 35 | 91.5 |
| 40 | 116 |
| 45 | 142.5 |
| 50 | 173 |
| 55 | 209.5 |
| 60 | 248 |
| 65 | 292.5 |
| 70 | 343 |
| 75 | 401 |
| 80 | 464 |

Source: U.S. Bureau of Public Roads

1. Graph (speed, average total stopping distance) using the data from the table. Tell which function family you think your graph belongs to. Justify your answer.
2. Graph each of the following and then write an equation for the graph.
   a. (speed squared, average total stopping distance)
   b. (speed cubed, average total stopping distance)
   c. (speed, the square root of average total stopping distance)

**c.** $\sqrt{d} = 0.25s + 0.965$, where $d$ = distance and $s$ = speed



## STUDENT PAGE 50

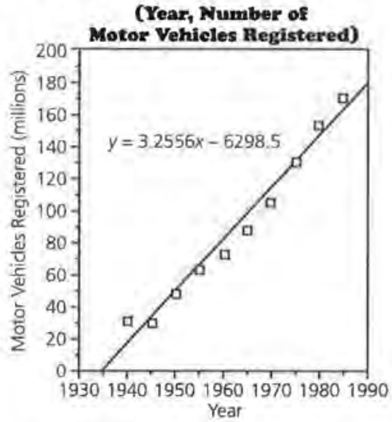
The number of motor vehicles registered every 5 years in the United States has increased since 1945.

| Year | Motor Vehicles Registered in U.S. (millions) |
|---|---|
| 1940 | 32.4 |
| 1945 | 31.0 |
| 1950 | 49.2 |
| 1955 | 62.7 |
| 1960 | 73.9 |
| 1965 | 90.4 |
| 1970 | 108.4 |
| 1975 | 132.9 |
| 1980 | 155.8 |
| 1985 | 170.2 |

Source: Moore & McCabe

3. Plot (year, motor vehicles registered in the U.S.).
   a. Draw the straight line of best fit through the data plot. Then find an equation for this line.
   b. The number of vehicles registered in 1945 does not fit the pattern. Explain why.
4. Identify a transformation that would straighten the curve. Create the table and plot the graph.
   a. Draw a straight line on the graph and find its equation.
   b. Which line do you think is better? Why?
5. Use the line you chose to predict how many motor vehicles will be registered in the year 2000.

**3.** **a.** $y = 3.2556x - 6298.5$

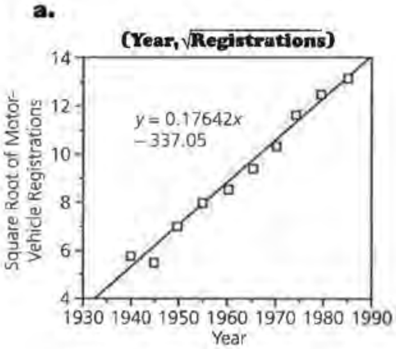



**b.** That was at the end of World War II, and the number of new automobiles had been severely restricted.

**4.** The transformation of square root of the registrations could possibly straighten the curve.

| Year | $\sqrt{\text{Registrations}}$ |
|---|---|
| 1940 | 5.692 |
| 1945 | 5.568 |
| 1950 | 7.014 |
| 1955 | 7.918 |
| 1960 | 8.597 |
| 1965 | 9.508 |
| 1970 | 10.412 |
| 1975 | 11.528 |
| 1980 | 12.482 |
| 1985 | 13.046 |

**a.**




**b.** Answers will vary. The second line will probably be chosen because the transformed data set appears to be straighter than the original.

**5.** Answers will vary. Using the line given above, $y = 0.17642(2000) - 337.05 = 15.79$. So there would be $(15.79)^2 \approx 249$ million registered motor vehicles in the year 2000.

UNIT III

# Mathematical Models from Data

## LESSON 7

# Transforming Data Using Logarithms

**Materials:** graph paper, rulers
**Technology:** graphing calculators or computer
**Pacing:** 2 class periods

### Overview

This lesson introduces students to the uses of logarithmic transformations in a formal manner. In previous lessons, the student has discovered the effects of many different transformations. In the ancestor problem they were informally (visually) shown the effect of applying a logarithmic transformation.

### Teaching Notes

The logarithm of a number is simply an exponent. The equations $b^y = x$, $b > 0$ and $b \neq 1$, is equivalent to $y = \log_b x$, $b > 0$, $b \neq 1$, and $x > 0$. In these equations, $y$ is the logarithm and $b$ is the base. Numbers may be compared by looking at their logarithms. Logarithms are often considered useful when numbers have magnitudes that are very large or very small.

This lesson assumes a knowledge of the definition and basic properties of a logarithm. In particular, Questions 7, 8, and 13 can use the property $\log_b xy = \log_b x + \log_b y$, $b > 0$, $b \neq 1$, $x > 0$, and $y > 0$. The lesson deals with application of the logarithmic transformation to data and its effects upon the graph's appearance. In Lesson 6, students were informally introduced to the idea of a logarithmic transformation when dealing with the ancestor problem and studied the effect if one simply used the exponent as the unit rather than the exponential function. In this lesson, students will be introduced to the effect of transforming the units of either axis using logarithmic transformations.

### Follow-Up

Students can be provided with, or asked to research, data sets whose scatter plots are nonlinear. They should then proceed to discover which transformation on that data would linearize its graph. They could be asked to conjecture which transformation they think would work and why, and then actually perform that transformation and either confirm or deny the validity of their conjecture. The next step would be to determine if they could conjecture the makeup of a data set that would require transformation by logarithms.

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LESSON 7

# Transforming Data Using Logarithms

**What is a logarithm?**

**How does a logarithmic transformation change the appearance of a graph?**

Units of measure such as meter, centimeter, foot, and inch increase consistently by 1s as they get larger. Therefore, a board that is 10 feet long is twice as long as a board that is 5 feet long. A meter stick is 10 times as long as a 10-cm ruler.

However, the number of ancestors in each generation does not increase steadily by 1s as you consider previous generations. Rather, the number increases by multiples of 2. Two generations ago, you had 4, or $2^2$, grandparents. Three generations ago, you had 8, or $2^3$, great-grandparents. Four generations ago, you had 16, or $2^4$, great-great-grandparents.

**OBJECTIVE**

Recognize how transforming either scale of a graph with the logarithmic function changes the shape of the graph.

**INVESTIGATE**

When numbers increase or decrease rapidly, it may be convenient to look for patterns in their exponents, or ***logarithms.***

| | |
|---|---|
| $\log_{10}10 = 1$ | The log of 10 equals 1 because $10^1 = 10$ and the log is the exponent. |
| $\log_{10}100 = 2$ | The log of 100 equals 2 because $10^2 = 100$ and the log is the exponent. |
| $\log_{10}1{,}000 = 3$ | The log of 1,000 equals 3 because $10^3 = 1{,}000$ and the log is the exponent. |

## Solution Key

## Discussion and Practice

1. Since $10^6 = 1{,}000{,}000$, the logarithm is 6.

2. Since any nonzero number raised to the zero power is 1, $10^0 = 1$ and the logarithm is 0.

3.

| Number | Power of 10 | Log$^{10}$ (Number) |
|---|---|---|
| 100,000,000,000,000 | $10^{14}$ | 14 |
| 10,000,000,000,000 | $10^{13}$ | 13 |
| 1,000,000,000,000 | $10^{12}$ | 12 |
| 100,000,000,000 | $11^{11}$ | 11 |
| 10,000,000,000 | $10^{10}$ | 10 |
| 1,000,000,000 | $10^9$ | 9 |
| 100,000,000 | $10^8$ | 8 |
| 10,000,000 | $10^7$ | 7 |
| 1,000,000 | $10^6$ | 6 |
| 100,000 | $10^5$ | 5 |
| 10,000 | $10^4$ | 4 |
| 1,000 | $10^3$ | 3 |
| 100 | $10^2$ | 2 |
| 10 | $10^1$ | 1 |
| 1 | $10^0$ | 0 |

4. $\text{Log}_{10}\ 0.1 = -1$, since $10^{-1} = 0.1$.

5. $\text{Log}_{10}\ 50$ is between 1 and 2, because $10^1 = 10$ and $10^2 = 100$.

6. $\log_{10}50 \approx 1.698970004$

7. $\text{Log}_{10}5$ is between 0 and 1, so $\log_{10}5 = \log_{10}50 - 1 \approx 0.698970004$.

8. $\text{Log}_{10}500$ is between 2 and 3, so $\log_{10}500 = \log_{10}50 + 1 \approx 2.698970004$.

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**Discussion and Practice**

1. What is $\log_{10}1{,}000{,}000$?
2. What is $\log_{10}1$? Why?
3. Complete the table.

| Number | Power of 10 | $\text{Log}_{10}$ (Number) |
|---|---|---|
| 100,000,000,000,000 | $10^{14}$ | |
| 10,000,000,000,000 | | |
| 1,000,000,000,000 | | |
| 100,000,000,000 | | |
| 10,000,000,000 | | |
| 1,000,000,000 | | |
| 100,000,000 | | |
| 10,000,000 | | |
| 1,000,000 | | |
| 100,000 | | |
| 10,000 | | |
| 1,000 | | |
| 100 | | |
| 10 | | |
| 1 | | |

4. What is $\log_{10}0.1$? Explain how you got your answer.
5. Use the table. Between what two numbers is $\log_{10}50$?
6. Use the $\boxed{\text{LOG}}$ button on your calculator to find $\log_{10}50$.
7. Between what two numbers is $\log_{10}5$? Use $\log_{10}50$ to find $\log_{10}5$.
8. Between what two numbers is $\log_{10}500$? Use $\log_{10}50$ to find $\log_{10}500$.
9. Is it possible for $10^n$ to be a negative number? Is it possible to find the $\log_{10}x$ if $x < 0$? Justify your answer.

9. It is not possible for $10^n$ to be negative. So it is not possible to determine the logarithm of a negative number.

**10.** $\log_2 1 = 0$

**11.** $\log_2 0.5 = -1$

**12.** $\log_2 64 = 6$

**13.** **a.** $\log_{10} 1 = 0$, $\log_2 1 = 0$

**b.** $\log_{10} 20 = \log_{10} 2 + \log_{10} 10 \approx 0.3 + 1 = 1.3$

$\log_2 20 = \log_2 10 + \log_2 2 \approx 3.3 + 1 = 4.3$

**c.** $\log_{10} 100 = 2$

$\log_2 100 = \log_2 10 + \log_2 10 \approx 3.3 + 3.3 = 6.6$

## STUDENT PAGE 55

Other bases may be used for logarithms. For example, in the ancestors problem, it is helpful to use base 2.

| | |
|---|---|
| $\log_2 2 = 1$ | The log base 2 of 2 equals 1. |
| $\log_2 4 = 2$ | The log base 2 of 4 equals 2. |
| $\log_2 8 = 3$ | The log base 2 of 8 equals 3. |

**10.** What is $\log_2 1$?

**11.** What is $\log_2 0.5$?

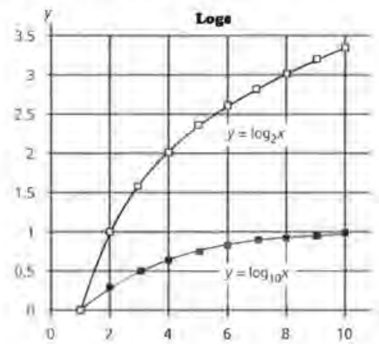
**12.** What is $\log_2 64$?

**13.** Use the graph below to find

**a.** $\log_{10} 1$ and $\log_2 1$.

**b.** $\log_{10} 20$ and $\log_2 20$.

**c.** $\log_{10} 100$ and $\log_2 100$.




**Summary**

The *logarithm* of a number is simply an exponent. The equations $b^y = x$ and $y = \log_b x$ are equivalent. In these equations, $y$ is the logarithm, and $b$ is the base. Numbers may be compared by looking at their logarithms. Logarithms are often considered useful when numbers have magnitudes that are very great or very small.

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## Practice and Applications

**14.** See table below.

**Practice and Applications**

**14.** Complete the rows on this table for each year. The logarithm of the debt can be abbreviated log (debt) when the base 10 logarithm is used.

| Year | Federal Debt | Federal Debt | Log (Debt) |
|---|---|---|---|
| 1980 | $909,100,000,000 | $9.091 \times 10^{11}$ | 11.9586 |
| 1981 | $994,900,000,000 | ______ | ______ |
| 1982 | $1,137,000,000,000 | ______ | ______ |
| 1983 | $1,372,000,000,000 | ______ | ______ |
| 1984 | $1,565,000,000,000 | ______ | ______ |
| 1985 | $1,818,000,000,000 | ______ | ______ |
| 1986 | $2,121,000,000,000 | ______ | ______ |
| 1987 | $2,346,000,000,000 | ______ | ______ |
| 1988 | $2,601,000,000,000 | ______ | ______ |
| 1989 | $2,868,000,000,000 | ______ | ______ |
| 1990 | $3,207,000,000,000 | ______ | ______ |
| 1991 | $3,598,000,000,000 | ______ | ______ |
| 1992 | $4,002,000,000,000 | ______ | ______ |
| 1993 | $4,351,000,000,000 | ______ | ______ |
| 1994 | $4,644,000,000,000 | ______ | ______ |
| 1995 | $4,921,000,000,000 | ______ | ______ |

The following table shows the cumulative number of different kinds of U.S. postage stamps issued, by 10-year intervals. This number includes only regular and commemorative issues and excludes such items as airmail stamps, special-delivery stamps, and postal cards.

| Year | Federal Debt | Federal Debt | Log(Debt) | Year | Federal Debt | Federal Debt | Log(Debt) |
|---|---|---|---|---|---|---|---|
| 1980 | $909,100,000,000 | $9.091 \times 10^{11}$ | 11.9586 | 1988 | $2,601,000,000,000 | $2.601 \times 10^{12}$ | 12.4151 |
| 1981 | $994,900,000,000 | $9.949 \times 10^{11}$ | 11.9978 | 1989 | $2,868,000,000,000 | $2.868 \times 10^{12}$ | 12.4576 |
| 1982 | $1,137,000,000,000 | $1.137 \times 10^{12}$ | 12.0558 | 1990 | $3,207,000,000,000 | $3.207 \times 10^{12}$ | 12.5061 |
| 1983 | $1,372,000,000,000 | $1.372 \times 10^{12}$ | 12.1374 | 1991 | $3,598,000,000,000 | $3.598 \times 10^{12}$ | 12.5561 |
| 1984 | $1,565,000,000,000 | $1.565 \times 10^{12}$ | 12.1945 | 1992 | $4,002,000,000,000 | $4.002 \times 10^{12}$ | 12.6023 |
| 1985 | $1,818,000,000,000 | $1.818 \times 10^{12}$ | 12.2596 | 1993 | $4,351,000,000,000 | $4.351 \times 10^{12}$ | 12.6386 |
| 1986 | $2,121,000,000,000 | $2.121 \times 10^{12}$ | 12.3265 | 1994 | $4,644,000,000,000 | $4.644 \times 10^{12}$ | 12.6669 |
| 1987 | $2,346,000,000,000 | $2.346 \times 10^{12}$ | 12.3703 | 1995 | $4,921,000,000,000 | $4.921 \times 10^{12}$ | 12.6921 |

# STUDENT PAGE 57

| Year | Cumulative Number of Kinds of U.S. Stamps Issued | Log (Number of Stamps Issued) |
|---|---|---|
| 1868 | 88 | ______ |
| 1878 | 181 | ______ |
| 1888 | 218 | ______ |
| 1898 | 293 | ______ |
| 1908 | 341 | ______ |
| 1918 | 529 | ______ |
| 1928 | 647 | ______ |
| 1938 | 838 | ______ |
| 1948 | 980 | ______ |
| 1958 | 1123 | ______ |
| 1968 | 1364 | ______ |
| 1978 | 1769 | ______ |
| 1988 | 2400 | ______ |

Source: *Scotts Standard Postage Stamp Catalog*, 1989

**15.** Find the log of the cumulative number of kinds of stamps for each year indicated.

**16.** Graph (year, cumulative number of kinds of stamps issued). Identify whether a linear or non-linear model best describes these data.

**17.** Graph (year, log(number of stamps issued)). Identify whether a linear or nonlinear model best describes these data.

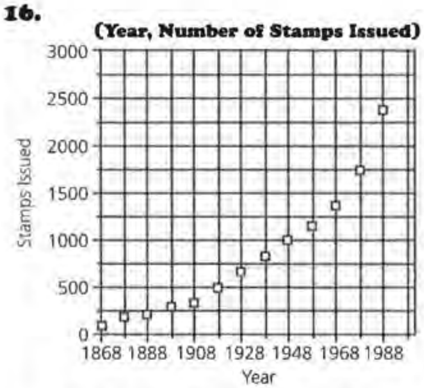
**18.** Describe the change that occurs on the graph when each $y$-value is transformed to log $y$.

**19.** Draw a line to model the (year, log(number of stamps issued)) graph, and write the equation of your line.

**15.**

| Year | Cumulative No. of Kinds of U.S. Stamps Issued | Log (Number of Stamps Issued) |
|---|---|---|
| 1868 | 88 | 1.9444827 |
| 1878 | 181 | 2.2576786 |
| 1888 | 218 | 2.3384565 |
| 1898 | 293 | 2.4668676 |
| 1908 | 341 | 2.5327544 |
| 1918 | 529 | 2.7234557 |
| 1928 | 647 | 2.8109043 |
| 1938 | 838 | 2.9232440 |
| 1948 | 980 | 2.9912261 |
| 1958 | 1123 | 3.0503798 |
| 1968 | 1364 | 3.1348144 |
| 1978 | 1769 | 3.2477278 |
| 1988 | 2400 | 3.3802112 |

**16.**



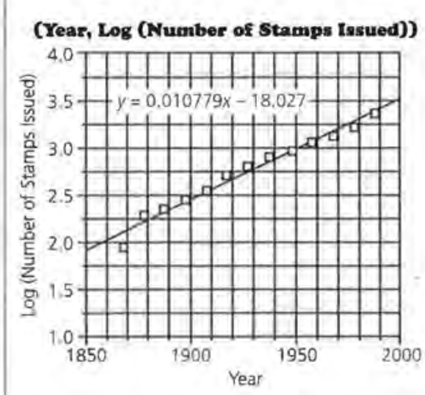

The graph is nonlinear and increasing, and it is possible that it could be modeled with a quadratic, cubic, or exponential model.

**17.**


The graph appears to be linear and increasing and could be modeled with a linear model.

**18.** When the $y$-values are replaced by log $y$ and each data point is plotted against the year, the graph becomes more linear.

**19.** $y = 0.0107787x - 18.0274$, where $y$ is the log of the number of U.S. stamps issued in the year $x$.



LESSON 8

# Finding an Equation for Nonlinear Data

**Materials:** graph paper, rulers, cup to shake coins, 100–200 pennies or other coins
**Technology:** graphing calculators or computer
**Pacing:** 2 class periods

## Overview

In this lesson, students will use a transformation to linearize the graph, determine the equation of the graph in its linear form, and then apply the inverse of that transformation to the equation to determine the equation of the original nonlinear data.

## Teaching Notes

Students must be encouraged to draw the line on the linearized data and determine its equation using the inverse transformation before they are introduced to or allowed to use the regression equation. It is assumed that the student has been introduced and is familiar with the laws of logarithms. This is a lesson on the application of logarithms and inverse transformations.

STUDENT PAGE 58

LESSON 8

# Finding an Equation for Nonlinear Data

**How can the application of an inverse function be used to find the model of the original data set?**

**What physical phenomenon can be modeled using an exponential function?**

Half-life is the amount of time it takes half of a substance to decay. Radioactive materials and other substances are characterized by their rates of decay, or decrease, and are rated in terms of their half-lives. The half-life of Carbon-14 allows scientists to date fossils. The half-life of a radioactive prescription substance is used to determine how frequently doses should be given.

**OBJECTIVE**

Find the equation of a nonlinear data set using transformations.

**INVESTIGATE**

Half-life can be simulated by the following experiment.

Equipment Needed: cup to shake coins; 100–200 pennies or other coins

Procedure: Put the coins into the cup. Shake the coins and pour them out. Remove all coins that land heads up. Record the number of coins removed and the number remaining on a chart like the one on page 59. Repeat this procedure until the number of coins remaining is 3, 2, or 1.

STUDENT PAGE 59

## Solution Key

## Discussion and Practice

**1.** 22,920 years

**2.** About 3.1 weeks

**3.** It is important for students to complete this activity. Groups of 3–4 usually work well, and each group needs a minimum of about 100 pennies. By performing the activity, students begin to realize how mathematical models can be used to describe natural patterns.

Answers will vary depending on number of coins and the circumstances of the shakes. We provide this as an example.

| Shake Number | Number of Coins Removed | Number of Coins Remaining |
|---|---|---|
| 0 | 0 | 100 |
| 1 | 48 | 52 |
| 2 | 26 | 26 |
| 3 | 15 | 11 |
| 4 | 7 | 4 |
| 5 | 4 | 1 |
| 6 | | |
| 7 | | |
| 8 | | |

| Shake Number | Number of Coins Removed | Number of Coins Remaining |
|---|---|---|
| 0 | 0 | original number |
| 1 | ____ | ____ |
| 2 | ____ | ____ |
| 3 | ____ | ____ |
| 4 | ____ | ____ |
| 5 | ____ | ____ |
| 6 | ____ | ____ |
| 7 | ____ | ____ |
| 8 | ____ | ____ |

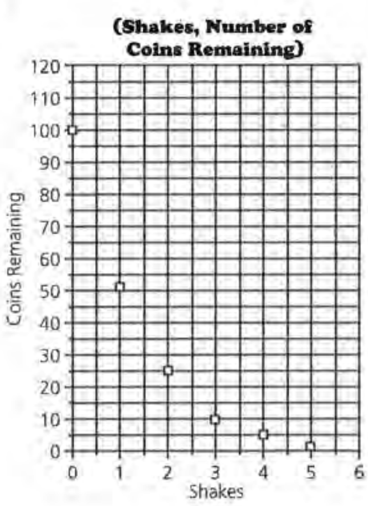
**Discussion and Practice**

1. The half-life of Carbon-14 is 5730 years. How long would it take for one gram of Carbon-14 to decay to $\frac{1}{16}$ of a gram?
2. The volume of a mothball, a small ball of napthalene used as a moth repellent, decreases by about 20% each week. What is the half-life of a mothball?
3. Perform the half-life simulation and record your results.
4. Graph the (shakes, number of coins remaining) ordered pairs from your experiment.
   a. Describe the pattern on the graph.
   b. Does the graph cross the $x$-axis? Where? What is the meaning of that point?
5. Approximately what percent of the remaining pennies were removed on each shake? Why? How many shakes represent a half-life for the pennies?

In order to decide what transformation best linearizes a curve, you may have to try more than one transformation.

6. Graph (shakes, square root of number of coins remaining). Draw the straight line that seems to best fit the graph.

**4.**

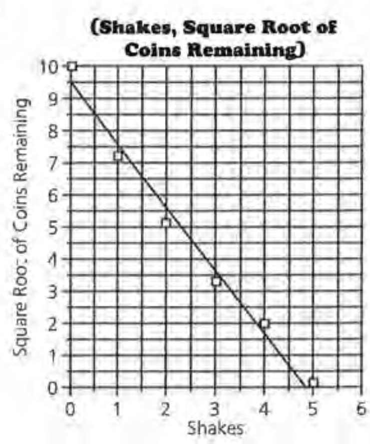



**a.** The curve is decreasing and nonlinear.

**b.** For this sample, the graph approaches the $x$-axis at the point (5, 0).

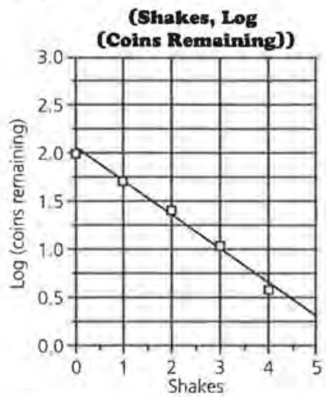
**5.** Approximately 50% of the coins remained after each shake. The probability of shaking a head or tail is 50%. One shake is the half-life of the pennies.

**6.**



# STUDENT PAGE 60

**7.** If a student records a zero for the last entry in the "Number of Coins Remaining" column, there will be a calculator error when the log transformation is applied. The zero can be changed to a positive number such as 1 to complete the modeling process.

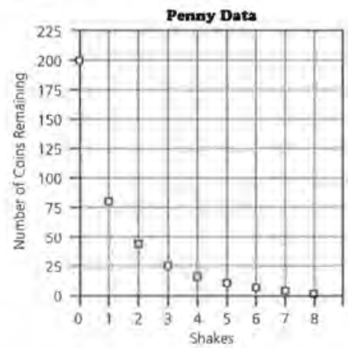



**8.** The second line appears to be the better line because each point appears closer to the line.

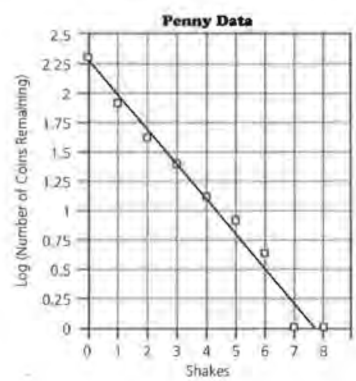
**7.** Graph (shakes, log(number of coins remaining)). Draw a straight line that seems to fit the graph best.

**8.** Compare the lines you drew for Questions 6 and 7. Which one do you think is better? Why?

The half-life simulation was performed by two students. Their data are graphed below.




The data were then transformed and a regression line was drawn as shown on the following graph.




The equation of the regression line in $y = ax + b$ form was $y = -0.298x + 2.275$.

STUDENT PAGE 61

9. a. The number of coins at the start

b. The fraction of the previous amount of coins left after each shake

10. Answers will vary since students have their own data. In our example, for (shakes, log (coins remaining)) the regression line is $y = -0.392x + 2.109$. Since $y$ is really the log of the number of coins remaining it will become:

$\log_{10} N = -0.392x + 2.109$,

where $N$ is the number of coins remaining after $x$ shakes.

Applying the definition of logarithm:

$N = 10^{-0.392x + 2.109}$

$N = (10^{-0.392x})(10^{2.109})$

$N = (10^{-0.392})^x)(10^{2.109})$

$N = (0.4055^x)(128.53)$

$N = (128.53)(0.4055^x)$

If students used a natural log transformation, then the natural log must be used in deriving the equation. The equation for the mathematical model in Question 10 can be graphed over the original data set to see how well the model fits the data.

The following example shows how this linear equation can be used to find the equation of the curve formed by the original data set. Study the example and make sure you understand what is happening.

Example: Using the Equation of a Regression Line to Find the Equation of the Original Data Set

Let the $x$- and $y$-variables in the ordered pairs $(x, y)$ represent points on the curve that contain the original data points. The equation of the regression line of the transformed data in the form $y = ax + b$ is written

$$\log_{10} y = ax + b,$$

since the transformation used $\log_{10} y$.

In this case, $a = -0.298$ and $b = 2.275$. For this regression line, $a$ is the slope and $b$ is the $y$-intercept.

Therefore, $\log y = -0.298x + 2.275$.

Applying the definition that $\log_b x = y$ implies $b^y = x$, it follows that:

$$y = 10^{(-0.298x + 2.275)}$$
$$y = (10^{-0.298x})(10^{2.275})$$
$$y = (10^{-0.298})^x (10^{2.275})$$
$$y = (0.504^x)(188.36)$$
$$y = 188.36(0.504^x)$$

9. This is a mathematical model for the data set shown in the first graph of Penny Data on page 60.

a. What does 188.36 represent?

b. What does 0.504 represent?

10. Find the equation of a mathematical model for the data you collected in your experiment.

**Summary**

Mathematical models are often used to answer questions or study trends. The process used in finding the equation may be summarized in the following steps.

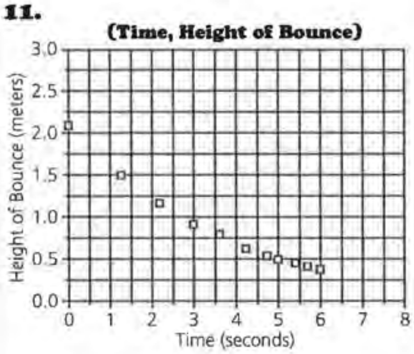
a. Collect and graph the data.

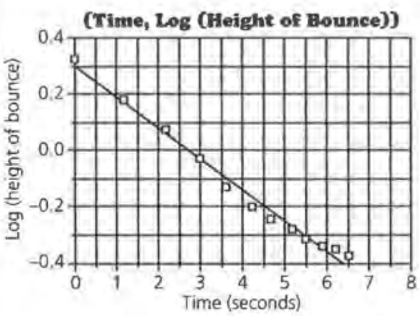
b. Observe patterns in the data.

c. If necessary, transform the data to straighten it. You may need to try more than one transformation.

## Practice and Applications

**11.**



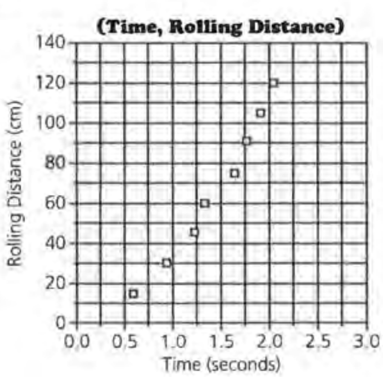

The regression line is $y = 0.305 - 0.1109x$. Since $y = \log$ (height of bounce), $\log(H) = 0.305 - 0.1109x$, where $H$ is the height at time $x$.

$H = 10^{0.305 - 0.1109x}$

$H = (10^{0.305})(10^{-0.1109x})$

$H = (2.018)(0.7746^{x})$

**12.**




# STUDENT PAGE 62

**d.** Plot the transformed data and draw a linear regression line.

**e.** Use the equation of the regression line to find an equation of the original data set.

*When a logarithmic transformation straightens a function, the function is an exponential function.*

**Practice and Applications**

In Questions 11 and 12, transform the data, if necessary, to find a linear model. Then use the linear model and inverse functions to find an equation for the original data.

**11.**

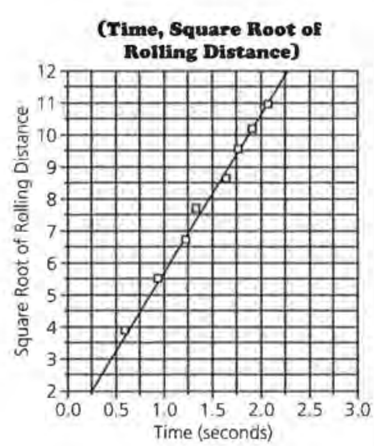
| Time (seconds) | Height of Bounce for Ball Dropped from 2.09 m (meters) |
|---|---|
| 0 | 2.09 |
| 1.2 | 1.52 |
| 2.2 | 1.2 |
| 3 | 0.94 |
| 3.6 | 0.76 |
| 4.2 | 0.63 |
| 4.7 | 0.57 |
| 5.2 | 0.52 |
| 5.5 | 0.49 |
| 5.9 | 0.45 |
| 6.2 | 0.44 |
| 6.5 | 0.42 |

Source: Physics Class, Nicolet High School

**12.**

| Distance for Metal Ball on Small Ramp at a Given Angle (cm) | Time (seconds) |
|---|---|
| 15.0 | 0.61 |
| 30.0 | 0.95 |
| 45.0 | 1.24 |
| 60.0 | 1.35 |
| 75.0 | 1.65 |
| 90.0 | 1.77 |
| 105.0 | 1.91 |
| 120.0 | 2.02 |

Source: Physics Class, Mahwah High School




The regression line equation is $y = 4.942x + 0.789$. Since $y$ = square root of the rolling distance, $\sqrt{D} = 4.942T + 0.789$, where $D$ is the rolling distance at time $T$.

$D = (4.942T + 0.789)^2$

$D = 24.42\,T^2 + 7.798\,T + 0.6225$

LESSON 9

# Residuals

**Materials:** graph paper, rulers
**Technology:** graphing calculators or computer
**Pacing:** 3 class periods

## Overview

This lesson introduces students to *residuals* and their value in the determination of how well a model fits a data set.

Residual = observed value – predicted value
Data = Fit + Residual

A mathematical model is used to describe a data set. Residuals are then determined and then used to describe the deviations of the data from the model. The residuals are plotted and studied for patterns. The patterns observed help in the search for a good mathematical model when those deviations are emphasized in the residual plot, $(x_i, \text{res}_i)$.

## Teaching Notes

It is important to have the students explore many lines fitted to data sets and the residuals that result. The patterns that are formed when the scatter plot is created by plotting $(x_i, \text{res}_i)$ are very informative once students learn what to look for in these plots. Allow students to conjecture and speculate about the patterns before they plot $(x_i, \text{res}_i)$. This lesson will analyze the residuals resulting from many of the examples students have been using in previous lessons. Then they will be required to create their own model, examine the residuals, and decide whether their model is a good fit.

## Follow-Up

Students can be provided with residual plots and asked to describe the patterns and discuss how well the mathematical model fits the data. Students can use problems from previous lessons and then use residuals to determine how well the model fits the data.

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LESSON 9

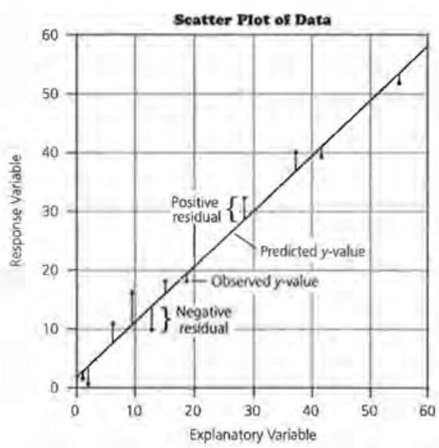
# Residuals

**What are residuals?**

**What do residuals reveal about the model?**

In many problems, you have been fitting a straight line to data in a scatter plot and evaluating the results to see if the line fits well. You have judged whether or not a fit was adequate mainly by the "eyeball" method: looking at the scatter plot, looking at the straight line in relation to the data points, and checking if the straight line makes sense as a model for the data. This method is important and is a good first step.

A numeric tool available to help in making this evaluation is a ***residual.*** The corresponding graphic tool is the ***residual plot.***




OBJECTIVE

Use plots of residuals to help assess how well a mathematical model fits the data.

## STUDENT PAGE 64

The residual may be calculated for each point in a data set. It is the difference between the actual or observed $y$-value and the predicted $y$-value found by using the mathematical model:

residual = observed $y$-value – predicted $y$-value from the model

Using symbols, $r_i = y_i - (a + bx_i)$.

### INVESTIGATE

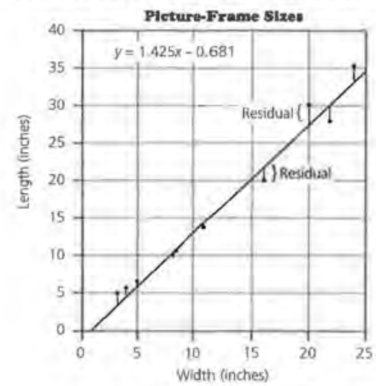
A fit, or predicted value, in the form $a + bx_i$ may be described by the symbol $\hat{y}_i$, read "y-hat sub $i$"; $\hat{y}_i = a + bx_i$:

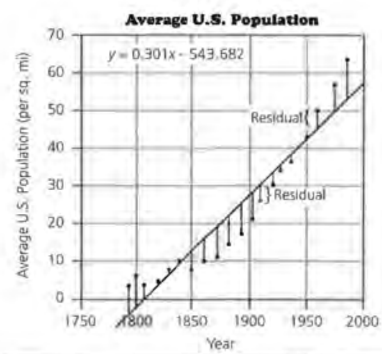
$$\text{residual}_i = \text{res}_i = y_i - \hat{y}_i$$

Also, observed data = fit + residual:

$$y_i = \hat{y}_i + (y_i - \hat{y}_i)$$

Consider the following scatter plots and residuals.



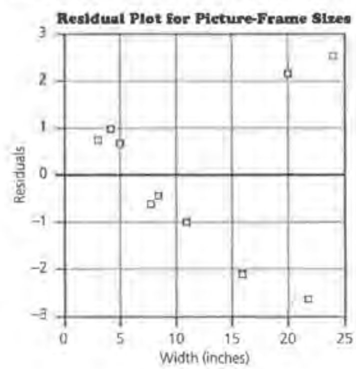



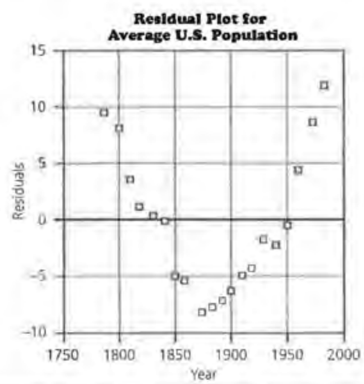
# STUDENT PAGE 65

When actual data are collected, a mathematical model generally does not fit them perfectly. It is not expected that the residuals' sum will be exactly zero. The residuals should, however, represent random variation of the data from the fitted model, some above and some below the horizontal line.

While residuals can be observed in an original scatter plot, it is sometimes helpful to make a separate plot of them to see if they form a pattern. You can do this by graphing $(x_i, y_i - \hat{y}_i) = (x_i, \text{res}_i)$, which is called a *residual plot* or plot of residuals against the explanatory variable $x$. In a good model, the residuals should exhibit a random pattern.

Consider the residual plots for the previous two graphs.



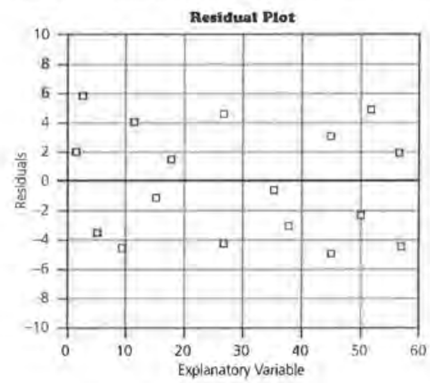



# STUDENT PAGE 66

In general, information from the residuals may indicate that the mathematical model is a good fit. This occurs when

- there are no obvious patterns in the relationship of residuals to the explanatory variable, $x$;
- there is uniform variability in the relationship of residuals to $x_i$; and
- there are no individual outlying points.

The residual plot should form an approximately uniform, horizontal band going across the page. This indicates a random pattern to the residuals with no special relationship to the explanatory variable $x$. The following graph shows an approximate horizontal band of data points somewhat uniform.




If the residuals from a mathematical model show a pattern with an obvious structure, the structure should be incorporated into the "fit" by changing the mathematical model, if possible. This can be done by transforming the data and fitting a new line. Then the residuals for the new model can be calculated, plotted, and observed. This process may continue several times for a data set.

## Solution Key

### Discussion and Practice

1. **a.** The data point is above the line.
   **b.** The data point is below the line.
   **c.** The data point is on the line.

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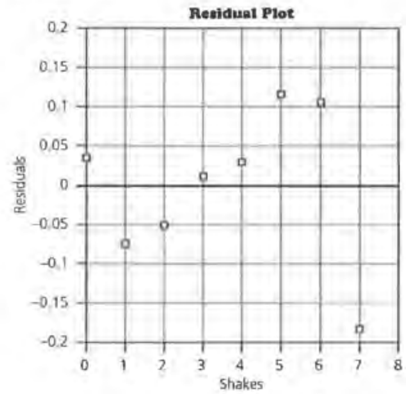
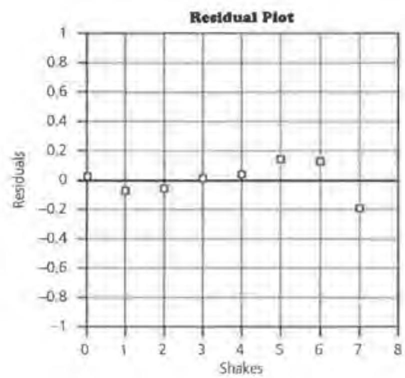
It is also important to consider the scale on the $y$-axis of the residual plot. The scale reveals the magnitudes of the residuals. The following plots of the same set of residuals for the Penny Data in Lesson 8 look different because the scales on the $y$-axes are different.




**Discussion and Practice**

1. Describe the vertical position on the graph of the actual or observed data point with respect to the value predicted by the mathematical model if the residual at the point is

   **a.** positive. **b.** negative. **c.** zero.

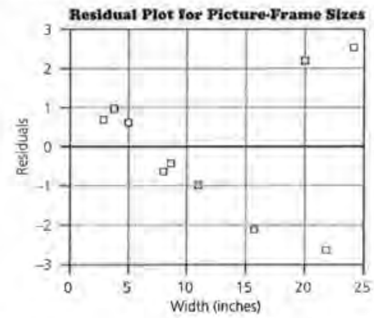
## STUDENT PAGE 68

2. Answers will vary. One method would be to sum the absolute values of the residuals and another is to plot the pairs ($x_i$, $res_i$).

3. The residuals have a definite pattern of moving farther from the zero line in both a positive and a negative direction as the width increases.

2. Describe at least two mathematical ways you could summarize the residuals for a mathematical model on a data set.

3. What patterns do you observe on the residual plot for Picture-Frame Sizes?




4. You have seen several graphs and the residual plot for the Average U.S. Population. The data set for these plots is given below.

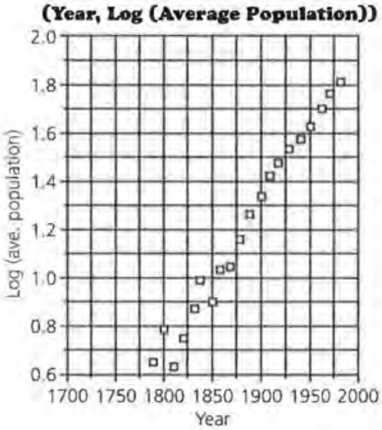
| Year | Average U.S. Population (per sq. mi) |
|---|---|
| 1790 | 4.5 |
| 1800 | 6.1 |
| 1810 | 4.3 |
| 1820 | 5.5 |
| 1830 | 7.4 |
| 1840 | 9.8 |
| 1850 | 7.9 |
| 1860 | 10.6 |
| 1870 | 10.9 |
| 1880 | 14.2 |
| 1890 | 17.8 |
| 1900 | 21.5 |
| 1910 | 26.0 |
| 1920 | 29.9 |
| 1930 | 34.7 |
| 1940 | 37.2 |
| 1950 | 42.6 |
| 1960 | 50.6 |
| 1970 | 57.5 |
| 1980 | 64.0 |

Source: *World Almanac and Book of Facts*, 1988

STUDENT PAGE 69

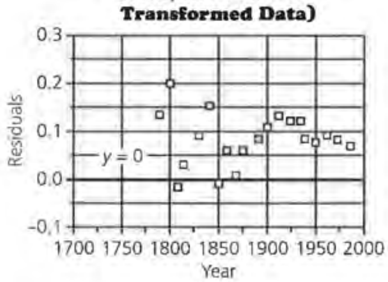
**4.** **a.** Have students note that a log transformation is being applied, where as in Lesson 5 a square-root transformation was applied to the same data.

**(Year, Log (Average Population))**



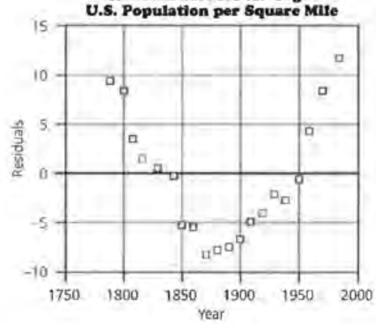

**b.** The equation of the regression line is $y = 0.0064x - 10.936$. Since $y$ represents log (average population), $\log P = 0.0064x - 10.936$, where $P$ is the average population in the year $x$. Adding three columns to the table yields the table on page 83.

**(Year, Residuals for Transformed Data)**

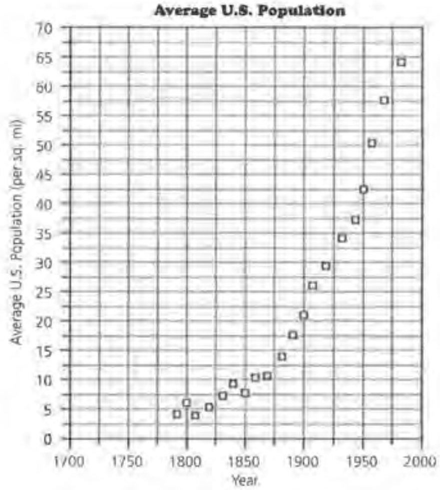



**c.** The new residual plot seems more random even though the majority of the residuals are positive and they seem to be decreasing as the year increases.

**Residual Plot for Average U.S. Population per Square Mile**




**Average U.S. Population**




**a.** The residual plot for average U.S. population shows a pattern that may be incorporated into a new model. Apply a transformation to one variable in the data set.

**b.** Find a mathematical model for the transformed data and make a residual plot.

**c.** Compare the previously given residual plot with the residual plot created in part b above. Decide which appears to have variation that is more random. Explain.

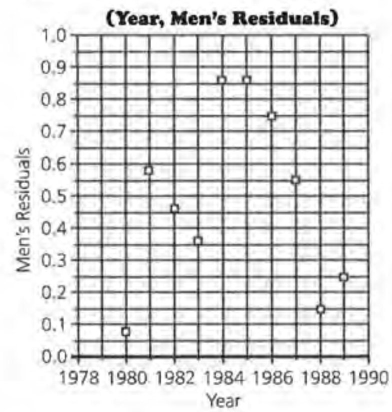
| Year | Average U.S. Population per Square Mile | Log of Average U.S. Population per Square Mile | Predicted Value from Regression Equation of Transformed Data | Residuals |
|---|---|---|---|---|
| 1790 | 4.5 | 0.653 | 0.52 | 0.133 |
| 1800 | 6.1 | 0.785 | 0.584 | 0.201 |
| 1810 | 4.3 | 0.633 | 0.648 | –0.015 |
| 1820 | 5.5 | 0.740 | 0.712 | 0.028 |
| 1830 | 7.4 | 0.869 | 0.776 | 0.093 |
| 1840 | 9.8 | 0.991 | 0.84 | 0.151 |
| 1850 | 7.9 | 0.898 | 0.904 | –0.006 |
| 1860 | 10.6 | 1.025 | 0.968 | 0.057 |
| 1870 | 10.9 | 1.037 | 1.032 | 0.005 |
| 1880 | 14.2 | 1.152 | 1.096 | 0.056 |
| 1890 | 17.8 | 1.250 | 1.16 | 0.090 |
| 1900 | 21.5 | 1.332 | 1.224 | 0.108 |
| 1910 | 26.0 | 1.415 | 1.288 | 0.127 |
| 1920 | 29.9 | 1.476 | 1.352 | 0.124 |
| 1930 | 34.7 | 1.540 | 1.416 | 0.124 |
| 1940 | 37.2 | 1.570 | 1.48 | 0.091 |
| 1950 | 42.6 | 1.629 | 1.544 | 0.085 |
| 1960 | 50.6 | 1.704 | 1.608 | 0.0966 |
| 1970 | 57.5 | 1.759 | 1.672 | 0.087 |
| 1980 | 64.0 | 1.806 | 1.736 | 0.070 |

# STUDENT PAGE 70

## Practice and Applications

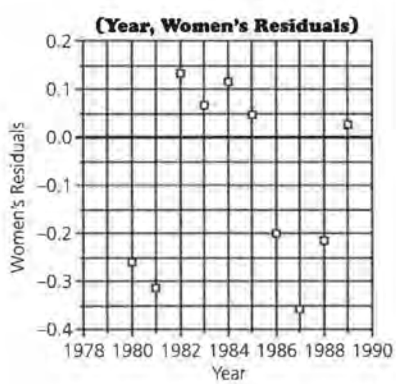
**5. a.**


The line of best fit is
$y = 1.00424x - 1968.789$.
The residuals are shown below.




**b.**


The line of best fit is
$y = 0.8569697x - 1685.036$.




**Summary**

A mathematical model can be used to describe a data set. Residuals describe the deviations of the data from the model.

residual = observed value – predicted value
data = fit + residual

Residuals may be plotted and studied for patterns. These patterns may help in the search for a good mathematical model because the deviations from the model are emphasized in the residual plot, $(x_i, res_i)$.

Use residuals in the modeling process using the following steps.

- Make a scatter plot of the data.
- Fit a line to the data.
- Calculate the residuals.
- Make a residual plot.
- Study the residual plot to see whether or not it exhibits random variation. If not, there may be a better mathematical model for the data.

**Practice and Applications**

**5.** Use the data in the table below to make scatter plots and residual plots for each of the following.

**a.** Median income of men over time

**b.** Median income of women over time

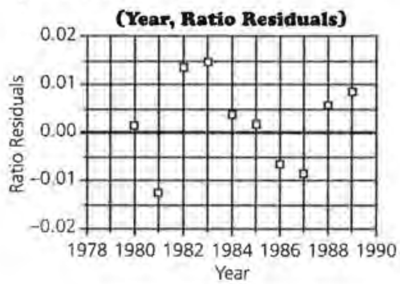
**c.** Ratio of median income of women to median income of men over time

**Median Income of Men and Women ($1,000)**

| Year | Men | Women |
|---|---|---|
| 1980 | 19.2 | 11.6 |
| 1981 | 20.7 | 12.4 |
| 1982 | 21.6 | 13.7 |
| 1983 | 22.5 | 14.5 |
| 1984 | 24.0 | 15.4 |
| 1985 | 25.0 | 16.2 |
| 1986 | 25.9 | 16.8 |
| 1987 | 26.7 | 17.5 |
| 1988 | 27.3 | 18.5 |
| 1989 | 28.4 | 19.6 |

c.

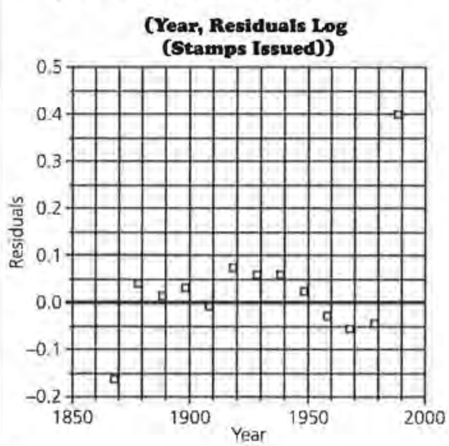

The line of best fit is
$y = 0.00878x - 16.78$.




6. In each of the graphs, the data are increasing and close to linear. In each case, the residual plots appear random.

7. Men's income begins higher and has a greater rate of change. Women's income will never catch up to the men's income at these rates.

8. From Question 17, Lesson 7,
$y = 0.0107787x - 18.0274$.

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6. What patterns do you notice in the scatter plots and the residual plots?

7. What observations can you make about median incomes of men and women?

8. Create a residual plot for the (year, log(number of stamps issued)) graph you created for Question 17 in Lesson 7.

| Year | Cumulative Number of Kinds of U.S. Stamps Issued | Log (Number of Stamps Issued) |
|---|---|---|
| 1868 | 88 | ——— |
| 1878 | 181 | ——— |
| 1888 | 218 | ——— |
| 1898 | 293 | ——— |
| 1908 | 341 | ——— |
| 1918 | 529 | ——— |
| 1928 | 647 | ——— |
| 1938 | 838 | ——— |
| 1948 | 980 | ——— |
| 1958 | 1123 | ——— |
| 1968 | 1364 | ——— |
| 1978 | 1769 | ——— |
| 1988 | 2400 | ——— |

Source: *Scotts Standard Postage Stamp Catalog*, 1989

LESSON 10

# Correlation: $r$ and $r^2$

**Materials:** graph paper, rulers
**Technology:** graphing calculators or computer
**Pacing:** 3 class periods

## Overview

In this lesson, students will be introduced to the numerical component in the analysis of how models fit data, ***correlation.*** The ***correlation coefficient*** may be useful in assessing how well a mathematical model fits a set of data. It is important to understand what the correlation coefficient measures and what it does not measure, how it can be interpreted, and what its properties and limitations are. Correlation is a measure of the strength of a linear relationship, or how tightly the points are packed around a straight line. The correlation coefficient is represented by the symbol $r$. The square of the correlation coefficient, represented by $r^2$, also has a useful interpretation.

The correlation coefficient, $r$, or its square, $r^2$, is often included in a statistical analysis with a least-squares line for a set of data. The formula for finding $r$ is generally programmed into a graphing calculator, and a calculator can be used to quickly and easily find $r$ for any pair of variables. On the TI-83, for instance, $r$ can be made to appear on the screen when the least-squares linear-regression line is calculated.

## Teaching Notes

The linear association between two variables can be assessed by a number, $r$, called the correlation coefficient. If there is perfect positive correlation, $r = 1$; if there is a perfect negative correlation, $r = -1$. A positive correlation indicates that as one variable increases the other also tends to increase while a negative correlation indicates that as one variable increases, the other tends to decrease. If $r$ is close to zero, then there is no good linear prediction of one variable from the other; knowing the value of one does not help you predict the other using a linear model. The correlation coefficient squared, $r^2$, indicates the proportion of error that can be explained by using the least-squares regression line. The closer $r^2$ is to 1, the more accurately $x$ can be used to predict $y$. The correlation coefficient measures only linear association rather than association in general. There may be a clear pattern in a set of data; but if it is not linear, the correlation may be close to 0. Correlation is a number without any units attached. Therefore, correlation does not depend on the units chosen for either variable. It is important, however, to look at the scatter plot to determine whether the relation is actually linear. People often confuse correlation with cause and effect. Just because two variables are correlated does not mean that one causes the other.

- They could both be a function of some other cause.
- One could cause the other.
- The relationship could be purely coincidental.

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LESSON 10

# Correlation: r and $r^2$

**What is correlation?**

**What additional information do $r$ and $r^2$ provide regarding the fit of a model?**

A numerical component in the analysis of how models fit data is *correlation*. ***Correlation*** is a measure of the strength of a linear relationship, or how tightly the points are packed around a straight line. The ***correlation coefficient*** is represented by the symbol $r$. The ***square of the correlation coefficient,*** represented by $r^2$, also has a useful interpretation. The correlation coefficient $r$ or its square, $r^2$, is often included in a statistical analysis with a least-squares line for a set of data. The formula for finding $r$ is generally programmed into a graphing calculator, and a calculator can be used to quickly and easily find $r$ for any pair of variables. On the TI-83, for instance, $r$ can be made to appear on the screen when the least-squares linear regression line is calculated.

The correlation coefficient may be useful in assessing how well a mathematical model fits a set of data. It is important to understand what the correlation coefficient measures, what it does not measure, how it can be interpreted, its properties, and its limitations. These topics will be investigated in this lesson.

Taken together, residual plots and the correlation coefficient can help assess how well a linear mathematical model fits a data set. These two mathematical tools also help in the selection of an appropriate model for a given data set.

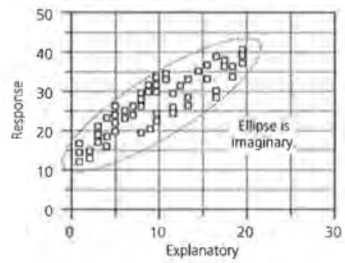
**OBJECTIVE**

Use the correlation coefficient and the square of the correlation coefficient along with residual plots to help assess how well a mathematical model fits a data set.

# STUDENT PAGE 73

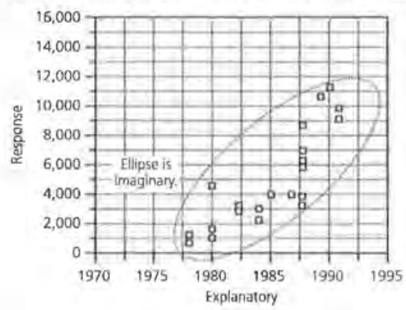
## INVESTIGATE

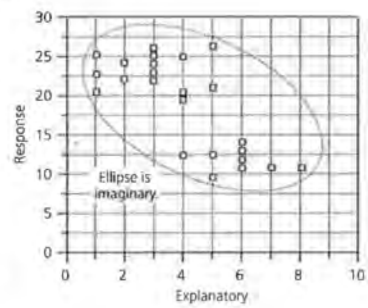
Consider the following scatter plot. If the data points are close to having a uniform spread throughout an imaginary ellipse enclosing all the data points, as in the plot shown below, it is reasonable to use correlation for measuring the association between these variables.




Imagine that you try to fit a line to the data set in the plot above. Do you see how tightly the points would be packed around the line you imagined?

If, however, the graph appears to have large gaps, empty areas, or a noticeable curved shape as in the next two plots, then correlation is not as useful for a measure of association.




Imagine lines through the data sets in these two plots. Do you see how tightly (or loosely) the points would be packed around the lines you imagined?

# STUDENT PAGE 74

It is often difficult to estimate the strength of the relationship between the explanatory and response variables from a plot. It is also difficult to meaningfully compare the degree of association in two different plots. A numerical measure of association is therefore useful. The correlation statistic is based on comparing how well $y$ can be predicted when $x$ is known to how well $y$ can be predicted when $x$ is not known. Some general properties of the correlation coefficient are listed below.

**Size**

- The value of $r$ always falls between $-1$ and 1. Positive $r$ indicates a positive association between the variables; that is, as $x$ increases, $y$ increases. Negative $r$ represents a negative association; that is, as $x$ increases, $y$ decreases.
- If $r = 0$, there is no linear relationship between the variables.
- The extreme values $r = -1$ and $r = 1$ occur only in the case of perfect linear association, when the points in the scatter plot lie exactly along a straight line.

**Units**

- The value of $r$ is not changed when the unit of measurement of $x$, $y$, or both $x$ and $y$ changes.
- The correlation $r$ has no unit of measurement; it is a dimensionless number.

**Linear Relation**

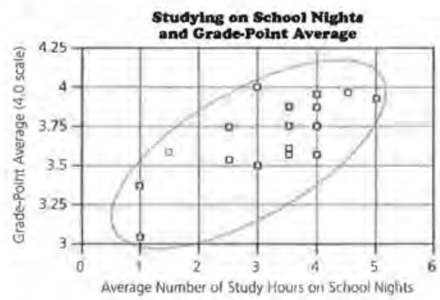
- Correlation measures only the strength of linear association between two variables.
- Curved relationships between variables, no matter how strong, are not reflected in the correlation.
- The square of the correlation coefficient, $r^2$, is the proportion of the variation in $y$ that can be explained by the variation in the value of $x$.

## STUDENT PAGE 75

Consider the relationship between grade-point average and the number of hours students study. Grade-point averages may vary from 0.0 to 4.0 on a 4-point scale. The correlation, $r$, of this data set is about 0.7, so the association is fairly strong.

| Number of Study Hours on School Nights | Grade-Point Average |
|---|---|
| 5 | 3.96 |
| 1 | 3.38 |
| 2.5 | 3.55 |
| 1.5 | 3.6 |
| 3.5 | 3.62 |
| 4 | 3.75 |
| 3 | 3.5 |
| 3.5 | 3.75 |
| 4 | 3.6 |
| 4.5 | 3.99 |
| 3 | 4.0 |
| 2.5 | 3.75 |
| 4 | 3.95 |
| 4 | 3.83 |
| 3.5 | 3.85 |
| 1 | 3.05 |
| 3.5 | 3.6 |

Source: Precalculus H Class, Nicolet High School




Since $(0.7)^2 = 0.49$, $r^2 = 0.49$. This means that 49% of the variability in grade-point averages can be explained by a linear relationship with how much the students study. However, 51% of the variability is unexplained or is due to other factors such as difficulty of classes, amount and quality of homework, and so on.

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## Solution Key

## Discussion and Practice

**1.** The values of $r$ fall between −1 and 1, inclusive.

**a.** If $r^2 = 0.81$, then $r$ has the possible value 0.9 or −0.9.

**b.** Students' plots will vary; however, positive correlation should have positive slope to the graph while negative correlation should show negative slope.

**c.** Students' sketches will vary; however, the points should form a nearly straight line.

**d.** Students' sketches will vary; however, they should display a random spread throughout the graph with no apparent linear appearance.

**2.** **a.** Answers will vary, but $r$ should be close to zero.

**b.** Answers will vary, but $r$ should be near 0.4.

**Discussion and Practice**

**1.** In general, what are the possible values for $r$?

**a.** If $r^2$ is 0.81, what are the possible values for $r$?

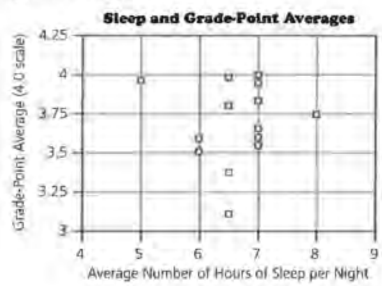
**b.** Sketch a scatter plot and a line corresponding to a positive value of $r$. Make a similar sketch for a negative value of $r$.

**c.** Sketch a plot showing a correlation close to 1.

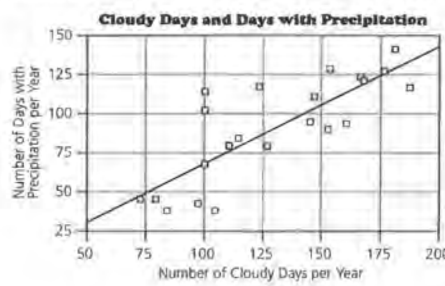
**d.** Sketch a plot showing a correlation close to zero.

**2.** Describe the correlation you would expect from looking at the following plots.

**a.** The hours of sleep on school nights versus the grade-point average.




**b.** The number of cloudy days per year in a set of cities versus number of days with precipitation per year



# STUDENT PAGE 77

**Correlation and Cause and Effect**

People often confuse correlation with cause and effect. Just because two variables are correlated does not mean that one causes the other.

- They could both be a function of some other cause,
- one could cause the other, or
- the relationship could be purely coincidental.

Consider the relation between overall grade-point averages and grades in English. The association is probably strong, but English grades alone do not cause high grade-point averages; other courses contribute also. The association between grade-point average and hours of study is high, and it is reasonable to assume that the time spent studying is a primary cause of grade-point averages. The correlation between grade-point averages and SAT scores is strong, but neither variable causes the other. A good SAT score does not cause high grade-point averages.

Sometimes the relationship occurs purely by chance. It could happen that the correlation between grade-point averages and the distances students live from school is strong. It seems unlikely that all the good students live the same distance from school. Much more reasonable is the assumption that the connection is coincidental, and there is no real link between distance from school and grade-point average.

There are several different kinds of correlation and different procedures for finding correlation between variables. The correlation coefficient described here is called Pearson's $r$, and it is the most commonly used type of correlation.

**Summary**

The linear association between two variables can be measured by a number $r$ called the correlation coefficient. If there is a perfect positive correlation, $r = 1$; if there is a perfect negative correlation, $r = -1$. A positive correlation indicates that as one variable increases, the other also tends to increase; while a negative correlation indicates that as one variable increases, the other tends to decrease.

If $r$ is close to zero, then there is no good linear prediction of one variable from the other; that is, knowing the value of one does not help you predict the other using a linear model.

## Practice and Applications

3. **a.** $r = 0.15$ and $r^2 = 0.0225$
   **b.** $r = 0.8$ and $r^2 = 0.64$
   **c.** $r = -0.8$ and $r^2 = 0.64$
   **d.** $r = 0.99$ and $r^2 = 0.9801$

# STUDENT PAGE 78

The correlation coefficient squared, $r^2$, indicates the proportion of error that can be explained by using the least-squares regression line. The closer $r^2$ is to 1, the more accurately $x$ can be used to predict $y$.

The correlation coefficient measures only linear association rather than association in general. There may be a clear pattern in a set of data, but if it is not linear, the correlation may be close to zero.

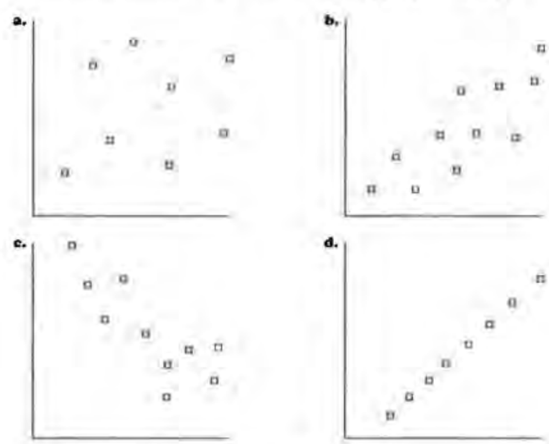
Correlation is a number without any units attached. Therefore, correlation does not depend on the units chosen for either variable.

Many software packages calculate $r$ automatically when they find the coefficients of the regression line. It is important, however, to look at the scatter plot to determine whether the relation is actually linear.

Correlation, the square of the correlation, and residual plots may be used to help assess a mathematical model or help select an appropriate mathematical model for data.

**Practice and Applications**

3. Match each correlation $r$ and $r^2$ with the appropriate graph.




$r = -0.8$ $r = 0.8$ $r = 0.99$ $r = 0.15$

$r^2 = 0.9801$ $r^2 = 0.0225$ $r^2 = 0.64$

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**4.** **a.** Answers will vary, but the sketch should not appear linear.

**b.** $r = 0.50$; see plot below.

**c.** The correlation coefficient for this graph does not support the article. It tells us only that the linear relationship is not strong.

**d.** The correlation of 0.5 implies that $r^2$ is 0.25, which indicates that 75% of the crime rate cannot be explained by this linear function.

**4.** The following quote appeared in a suburban Milwaukee newspaper article with the title *Spending More on Police Doesn't Reduce Crime.*

A CNI study of crime statistics and police department budgets over the last four years reveals there really is no correlation between what a community spends on law enforcement and its crime rate.

**a.** Sketch what you think a plot of the data would look like based on the quote from the article.

**b.** Use the data below about the suburban crime rate and the per-capita spending on police. Plot the data and find the correlation coefficient.

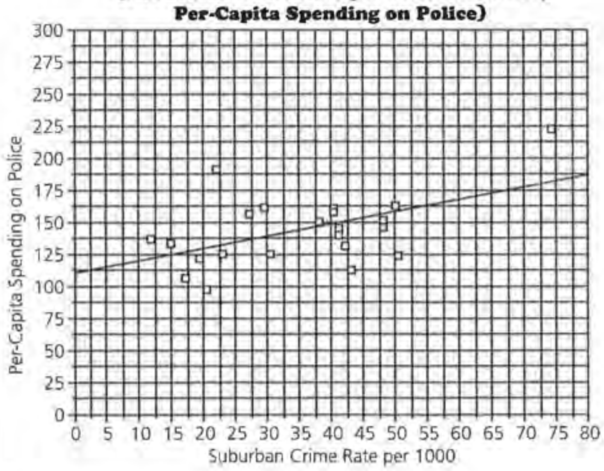
| Community | Suburban Crime Rate per 1,000 Residents | Per-Capita Spending on Police |
|---|---|---|
| Glendale | 74.39 | \$222.25 |
| West Allis | 50.43 | \$164.47 |
| Greendale | 50.25 | \$123.43 |
| Greenfield | 48.68 | \$143.59 |
| Wauwatosa | 48.52 | \$150.34 |
| South Milwaukee | 43.20 | \$110.64 |
| Brookfield | 42.16 | \$131.42 |
| Cudahy | 41.47 | \$137.12 |
| St. Francis | 41.32 | \$144.84 |
| Shorewood | 40.84 | \$156.39 |
| Oak Creek | 40.84 | \$160.41 |
| Brown Deer | 37.09 | \$150.57 |
| Germantown | 31.16 | \$125.86 |
| Menomonee Falls | 29.73 | \$159.28 |
| Hales Corners | 27.04 | \$155.40 |
| New Berlin | 25.45 | \$125.33 |
| Franklin | 23.09 | \$ 94.92 |
| Elm Grove | 21.86 | \$191.64 |
| Whitefish Bay | 21.28 | \$120.34 |
| Muskego | 17.00 | \$105.35 |
| Fox Point | 15.07 | \$132.85 |
| Mequon | 11.31 | \$136.39 |

Source: *Hub*, November 4, 1993

**c.** Does the correlation coefficient support the conclusions in the paragraph?

**d.** What does $r^2$ indicate about the relation between spending and the crime rate?

**(Suburban Crime Rate per 1000 Residents, Per-Capita Spending on Police)**



## LESSON 11

# Developing a Mathematical Model

**Materials:** graph paper, rulers
**Technology:** graphing calculators or computer
**Pacing:** 3 class periods

### Overview

This lesson is devoted to the application and practice of all the procedures students have become familiar with in Lessons 1 through 10.

### Teaching Notes

It must be understood that the procedure used in this lesson is the development of a mathematical model. The concept of modeling is to use mathematical and statistical tools and techniques to create an equation or model to better understand a more complex process. If more sophisticated tools or additional data become available, the model can be changed to incorporate the new information. The interaction between mathematical models and data continues as long as man increases his knowledge.

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LESSON 11

# Developing a Mathematical Model

**What is a mathematical model?**

**What mathematical concepts can be used to determine the best model for a given data set?**

The Tree Growers Association has collected data about the ages of chestnut oak trees and their respective trunk sizes as measured by diameter. The Association would like to know if there is an optimum time to harvest the trees.

OBJECTIVE

Use the knowledge of transformations, logarithms, residuals, and correlation to develop a mathematical model.

INVESTIGATE

Because of your ability to develop mathematical models, you have been selected to prepare a report to be delivered at the next monthly meeting of the Tree Growers Association. Your task in this lesson includes two assignments.

- Find a mathematical model for the relationship between age and the size of the diameter of chestnut oak tree trunks using the data on page 81.
- Prepare a report that explains how you developed the model.

Your model will be used to help determine an optimum time to harvest the trees.

## Solution Key

### Discussion and Practice

1. Answers will vary. The age column is increasing but not at a constant rate. There is very little pattern noticeable in the diameter column, except that there seems to be an increasing but nonconstant pattern. The pattern that seems evident in the two columns is: as the age increases the diameter increases, or as the diameter increases the age increases.

2. Since at one time the increase in diameter is 1.5 inches per year and at another time there is a decrease in diameter, the change is not constant as age increases.

3. Answers will vary. One pattern is that they are both increasing at nonconstant rates.

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Chestnut Oak Trees

| Age (years) | Trunk Diameter (Inches) |
|---|---|
| 4 | 0.8 |
| 5 | 0.8 |
| 8 | 1 |
| 8 | 2 |
| 8 | 3 |
| 10 | 2 |
| 10 | 3.5 |
| 12 | 4.9 |
| 13 | 3.5 |
| 14 | 2.5 |
| 16 | 4.5 |
| 18 | 4.6 |
| 20 | 5.5 |
| 22 | 5.8 |
| 23 | 4.7 |
| 25 | 6.5 |
| 28 | 6 |
| 29 | 4.5 |
| 30 | 6 |
| 30 | 7 |
| 33 | 8 |
| 34 | 6.5 |
| 35 | 7 |
| 38 | 5 |
| 38 | 7 |
| 40 | 7.5 |
| 42 | 7.5 |

Source: *Elements of Forest Mensuration*, Chapman and Demeritt

**Discussion and Practice**

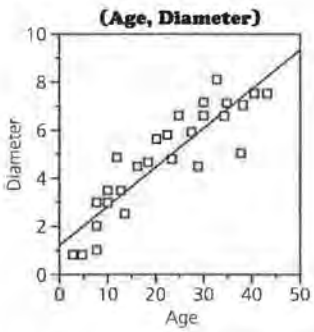
1. Identify any patterns you observe when looking at the two columns of data individually. Consider the amount of increase within a column and the range of values.
2. Does the increase in diameter appear to be constant as age increases?
3. What patterns do you notice in the dependence between variables?

4. Since the age was known and the tree diameter determined, the ordered pair is (age, diameter). The explanatory variable is age; the response variable is diameter.

5. Answers will vary. The scatter plot shows an increasing pattern. It is nonlinear but does not clearly indicate a recognizable curve.

6. Not very well; $r = 0.89$, but the graph of the data points appears curved.

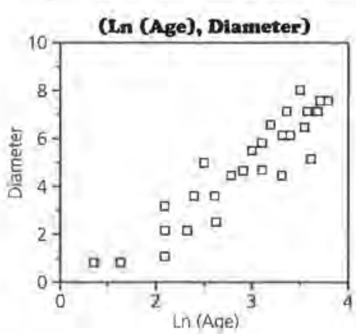
a.




b. The line does not appear to capture the characteristics of the data. The data appear to form some sort of curve, which is evidenced by the fact that the majority of the points in the center are above the line while the points below the line seem to be predominantly on either end.

7. Answers will vary. One transformation could be to square the diameter and another to take the logarithm of the age.

8. Sample:




## STUDENT PAGE 82

4. The Tree Growers Association reported they had recorded when the oak trees were planted; this fact made the data on their ages available. Determine the explanatory variable and the response variable for the data set. This identification will assist in determining the order in the ordered pairs.

**Plot the Data**

5. Enter the data into a graphing utility and draw a scatter plot. Write a short paragraph to identify the characteristics of the scatter plot.

6. Does a linear model fit the data?

   a. Draw a straight line that fits the scatter plot.

   b. Observe whether the line captures the characteristics you identified. Support your claim with a written argument.

**Transform the Data to Straighten the Scatter Plot**

To help you decide what transformations may straighten the data,

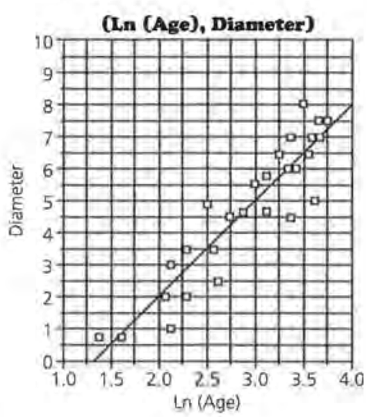
- use your knowledge of graph patterns for different functions, and
- consider what you know about the relationship between the explanatory and response variables.

7. Identify at least two transformations that you think will straighten the scatter plot of the original data set. Write a short paragraph explaining why you chose those transformations and to which variables you would apply them.

8. Make a scatter plot of each set of transformed data you think appears linear when graphed. Label the axes properly.

9. Use appropriate technology to draw the linear regression line on each graph. Then find and record the equation for each regression line and the correlation coefficient.

9. Sample:




The equation of the line is diameter = ln (age) – 3.973 with a correlation coefficient of 0.92.

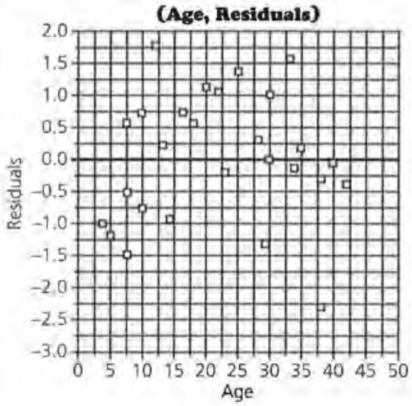
**10.** Sample:

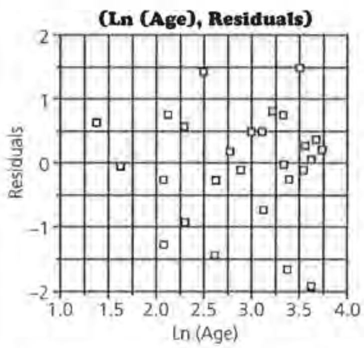
**a.** $3(\ln(33)) - 3.97 \approx 6.5$ inches

**b.** $3(\ln(21)) - 3.97 \approx 5.2$ inches

**c.** $3(\ln(60)) - 3.97 \approx 8.3$ inches

**11.** Sample:




Answers will vary with students' choice of transformation. Example:

- There seems to be much more of a pattern in the first set of residuals than the second. The first looks like an inverted parabola.
- It does not appear that either plot contains a point that has undue influence.
- No; they both seem to be more random than suggested in this question. However, the first plot does seem to have the pattern talked about in the answer to part a.

## STUDENT PAGE 83

**10.** Use the lines you chose to predict the diameter of a tree for these specific ages.

**a.** 33 years

**b.** 21 years

**c.** 60 years

**Compare One Transformation to Another**

In order to determine which transformation is better, it is helpful to consider two different statistical tools: residuals and correlation.

Following are some questions to consider when looking at the plots of residuals.

- Do the residuals reveal a pattern that can be used to predict the error? If so, the line may not be considered a good fit for the data. It is better to have a random distribution of points.
- Do one or more of the data points have more influence that they should on the regression line? If so, examine the data points again.
- Do the residuals have a narrow vertical spread at one end of the plot and a wider spread at the other end? It is better to have a constant variation in the spread across the values of the explanatory variable (domain).

**11.** Use appropriate technology to draw residual plots for two of the scatter plots and lines you drew in Questions 8 and 9. Then discuss the three questions above for your residual plots.

**12.** Compare the correlation coefficients and the squares of the correlation coefficients for the lines you drew in Questions 8 and 9. Explain what they tell you about the data and the mathematical model.

**13.** Use what you learned by looking at residuals and correlation to choose a mathematical model for the oak trees data set.

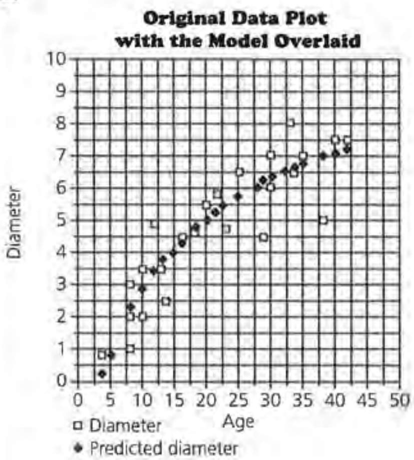
**12.** Answers will vary. For the (age, diameter) plot, $r^2 = 0.80$ and $r = 0.89$. For the (ln(age), diameter) plot, $r^2 = 0.85$ and $r = 0.92$. This implies that the second plot is a little better, since the residuals do not have a definite pattern as well.

**13.** Answers will vary. In our example, the better choice is the ordered pair (ln (age), diameter).

**14.** Answers will vary. For this example, diameter = 3 ln(age) – 3.97.

**15.** Answers will vary. For this example, the equation is diameter = 3 ln(age) – 3.97.

**16.**




It appears that the model is a good fit.

**17.** Answers will vary. The summary of how students developed their model should contain some discussion of what factors entered into their decision to choose the specific model. The recommendation to the Tree Growers Association will be a personal statement but should contain evidence gathered using their model and the information gained in this unit.

## STUDENT PAGE 84

**Find a Model for the Original Data Set**

**14.** Use the model you selected in Question 13. Rewrite each equation replacing the explanatory and response variables with the transformed variables. Remember that this equation is linear.

**15.** Use your knowledge of algebra and inverse functions to find a mathematical model for the original data set.

**16.** Graph the chestnut oak trees data and your model on the same graph. Describe your observations.

**Interpret Results and Write a Summary**

**17.** Write a summary report. Include the following.

**a.** A summary of how you determined your model.

**b.** A model (function) for the original data set.

**c.** A graph that contains the original data set and the model.

**d.** Your recommendation for the Tree Growers Association.

**Summary**

*Process for Finding a Mathematical Model*

- Study the data and identify patterns.
- Make a scatter plot of the data and examine the plot for any patterns.
- Look for functional relationships and try one or more transformations to straighten the scatter plot. Find linear models for the transformed data.
- Use residuals and correlation to assist in determining the best transformation for linearizing the data.
- Use your knowledge of transformations and functions to generate a model of the original data set.
- Interpret the results and write a summary of your findings.

# STUDENT PAGE 85

It must be understood that the procedure used in this lesson is the development of a mathematical model. The concept of modeling is to use mathematical and statistical tools and techniques to create an equation or model to better understand a more complex process. If more sophisticated tools or additional data become available, the model can be changed to incorporate the new information. The interaction between mathematical models and data continues as long as man increases his knowledge.

**Practice and Applications**

**18.** In 1981, Boeing Aircraft Company did a study of people and their fear of flying. Data were obtained from a simple survey question, "Are you afraid of flying?" with responses of "yes" or "no" and the person's age.

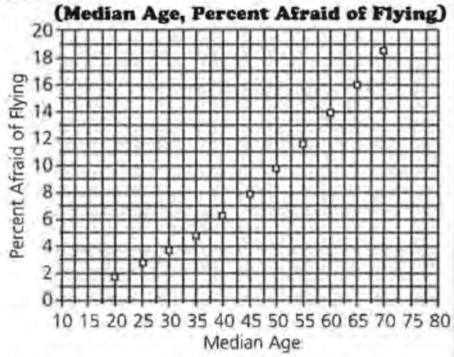
| Median Age | Percent of Population Sampled Afraid of Flying |
|---|---|
| 20 | 1.840 |
| 25 | 2.670 |
| 30 | 3.690 |
| 35 | 4.890 |
| 40 | 6.280 |
| 45 | 7.850 |
| 50 | 9.610 |
| 55 | 11.550 |
| 60 | 13.680 |
| 65 | 15.990 |
| 70 | 18.490 |

**a.** Create a model to describe the relationship between median age and the percent afraid of flying.

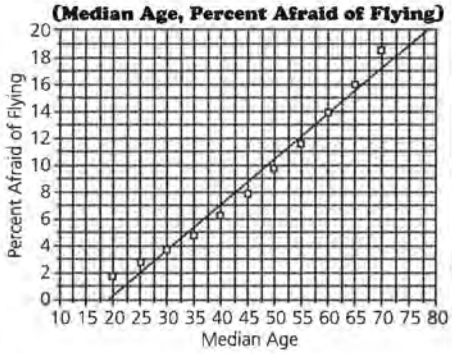
**b.** Prepare an argument defending your model.

**18.**

**(Median Age, Percent Afraid of Flying)**

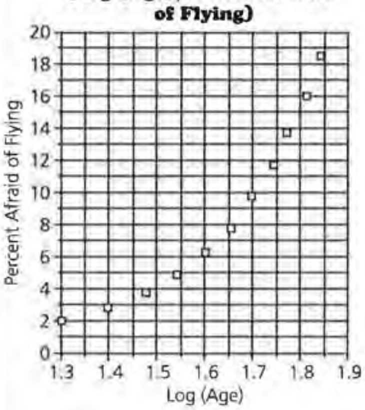



**(Median Age, Percent Afraid of Flying)**



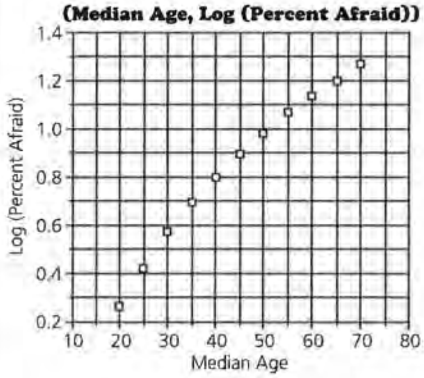

Fitting a straight line to the data underlines the fact that the data are not linear. Some transformations that may be tried are (log (age), percent afraid), (age, log (percent afraid), and (age, $\sqrt{\text{percent afraid}}$). Here are some results of these transformations with analysis.

**(Log (Age), Percent Afraid of Flying)**

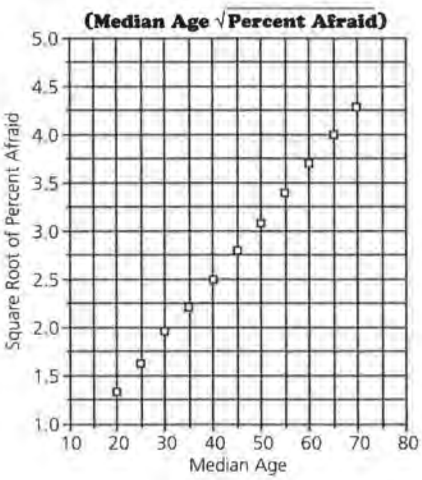



This transformation seems to compound the curvature. Probably the wrong variable was transformed.

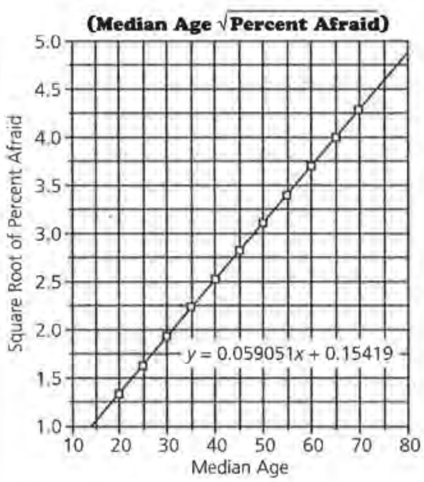
**(Median Age, Log (Percent Afraid))**




This transformation has only reversed the curvature.




This transformation seems to linearize the data. Create a regression line and plot it on this graph.



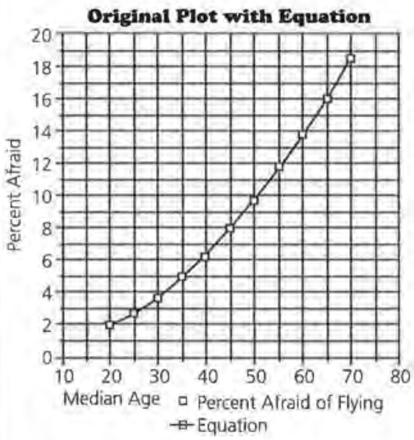

The regression equation is $y = 0.059\,x + 0.154$ with a correlation coefficient of 0.99995.

Knowing that $y = \sqrt{\text{percent afraid}}$, letting $F$ = percent afraid and $x$ = median age implies $\sqrt{F} = 0.059x + 0.154$.

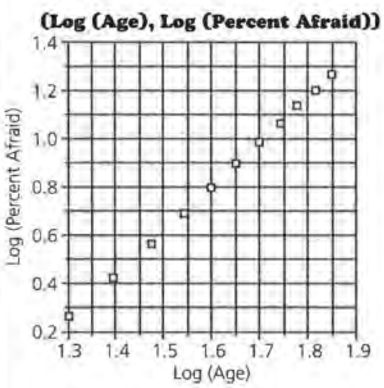
Hence, $F = (0.059x + 0.154)^2 = 0.0035x^2 + 0.018x + 0.024$.

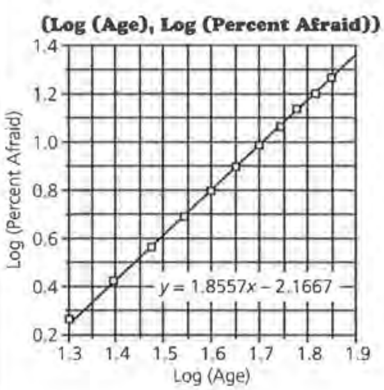
This is the equation of the original data set.

Superimposing the graph of this equation on the original graph yields the following :




NOTE: This is not the only solution to this problem. Consider, for example, plotting (log(age), log(percent afraid)).




The regression equation $y = 1.86x - 2.17$ implies

$\log(F) = 1.86 \log(A) - 2.17$,

where $F$ = the percent afraid of flying at age $A$.

Applying the inverse,

$F = 10^{1.86 \log(A) - 2.17}$

$= 10^{1.86 \log(A)} \bullet 10^{-2.17}$

$= 10^{\log(A^{1.86})} \bullet 10^{-2.17}$

$= 0.0676(A^{1.86})$.

By definition of logarithm,

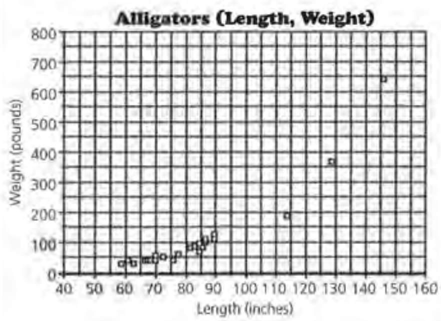
$F = 0.00676(A)^{1.86}$.

This also fits the original data.

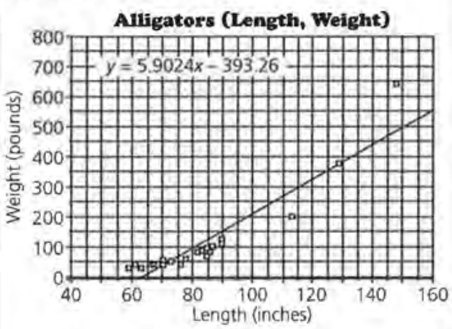
To further delineate between the two transformations, plot the residuals and consider the relative sizes of those residuals.

### Solution Key

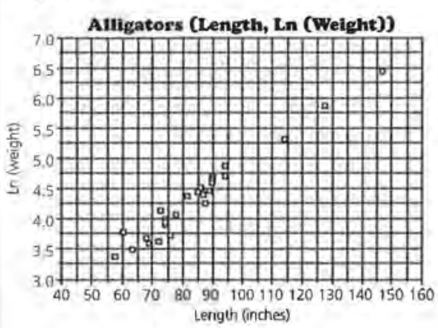
Answers will vary because the choice of transformation and the assessment of fit will be personal judgments. An example is provided to guide you in your effort to assess students' work.




This scatter plot appears to be nonlinear and increasing.




The correlation coefficient of the regression line has a value of 0.91, but the shape of the data points leads one to suspect that there is likely a transformation that will have a better regression.




This transformation seems to have linearized the data very well.

## STUDENT PAGE 86

PROJECT

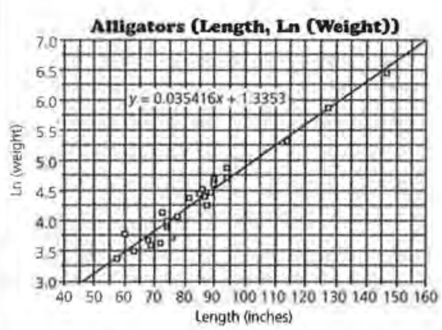
# Alligators' Lengths and Weights

The following table of data was created by the Florida Game and Freshwater Fish Commission. Your assignment is to determine what function could be used to model the relationship between the length and weight of an alligator, or whether such a relationship even exists. Prepare a formal presentation (charts, graphs, and numeric and symbolic arguments), utilizing all the processes you have studied in this module, in defense of your position.

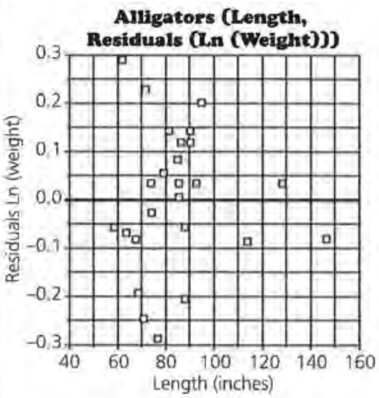
| Length (in.) | Weight (lb) | Length (in.) | Weight (lb) |
|---|---|---|---|
| 94 | 130 | 86 | 83 |
| 74 | 51 | 88 | 70 |
| 147 | 640 | 72 | 61 |
| 58 | 28 | 74 | 54 |
| 86 | 80 | 61 | 44 |
| 94 | 110 | 90 | 106 |
| 63 | 33 | 89 | 84 |
| 86 | 90 | 68 | 39 |
| 69 | 36 | 76 | 42 |
| 72 | 38 | 114 | 197 |
| 128 | 366 | 90 | 102 |
| 85 | 84 | 78 | 57 |
| 82 | 80 | | |

Your report must include

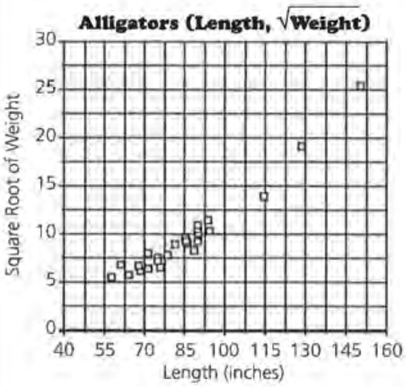
- scatter plots of original and transformed data with the patterns in those graphs identified in writing,
- residual plots used and an analysis of each plot,
- correlation coefficients and related conclusions, and
- equations of lines, including an equation that can be used with the original data as a predictor equation.



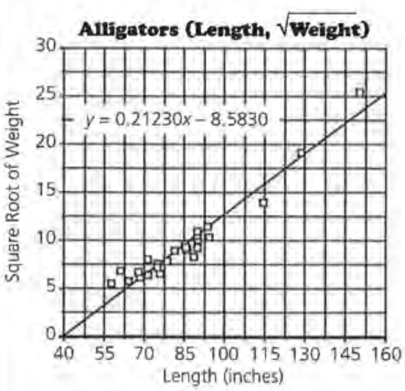

The drawing of the linear regression line with equation ln (weight) = 0.035416 (length) + 1.3353 confirms that the transformed data are relatively linear. The correlation coefficient of 0.98 and $r^2 = 0.96$ support the conclusion very well.




Plotting the residuals from the ln(weight) against the length, it appears that the residuals are quite random.



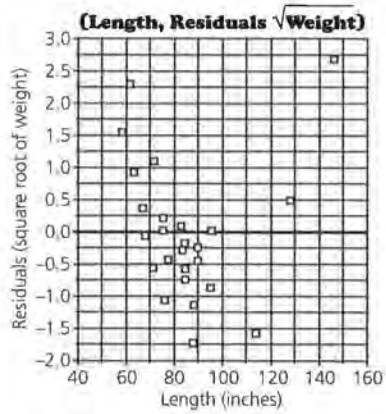

While this transformation does appear to linearize the data, it does not appear to do it as well as the natural log transformation.




The equation of the regression line is $\sqrt{\text{weight}} = 0.2123(\text{length}) - 8.583$.

$r^2$ is approximately 0.94, implying that $r$'s value would be 0.97. These values, while good, do not seem to be as good as the natural log transformation.

Furthermore, plotting the residuals of the $\sqrt{\text{weight}}$ against the length shows that there does seem to be a pattern in the residuals.




With all this information, it can be concluded that an equation for the line of best fit is

$\ln(W) = 0.035416\,L + 1.3353$,

where $W$ is the weight and $L$ is the length.

Using the properties of logarithms,

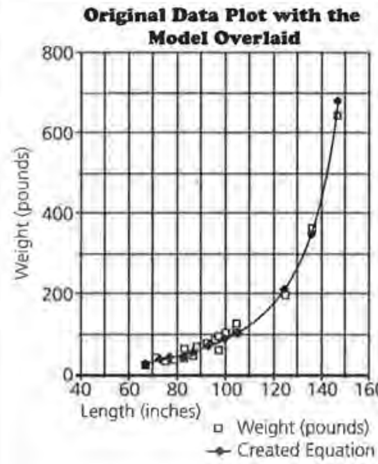
$W = e^{(0.035416\,L + 1.3353)}$

$= e^{1.3353}(e^{0.035416L})$

$= e^{1.3353}(e^{0.035416})^{L}$

$= 3.8011(1.036L)$.

When this equation is plotted on the original graph, there is a very good fit.



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## ASSESSMENT

# The Growth of Bluegills

The following table of data concerning the length of bluegills was created by the Florida Game and Freshwater Fish Commission.

| Initial Length (mm) | Total Length After 1 Year (mm) | Initial Length (mm) | Total Length After 1 Year (mm) |
|---|---|---|---|
| 48 | 69 | 138 | 160 |
| 52 | 71 | 138 | 157 |
| 51 | 69 | 130 | 156 |
| 53 | 75 | 140 | 161 |
| 69 | 101 | 160 | 173 |
| 71 | 107 | 157 | 168 |
| 69 | 100 | 156 | 172 |
| 75 | 104 | 161 | 178 |
| 101 | 138 | 173 | 176 |
| 107 | 138 | 168 | 174 |
| 100 | 130 | 172 | 173 |
| 104 | 140 | 178 | 178 |

1. Make a scatter plot of these data.
2. Identify in a short paragraph the characteristics you determine from the numeric and graphic displays of these data.
3. Draw a straight line through the scatter plot and determine whether or not the line captures any or all of the characteristics you identified in Question 2.
4. If the data set's scatter plot does not appear linear, perform a series of transformations on one or both of the variables to attempt to linearize the plot.
5. Select the plot or plots that appear the most linear, find the linear model, and check $r$ and $r^2$ to determine how well the model fits the data.

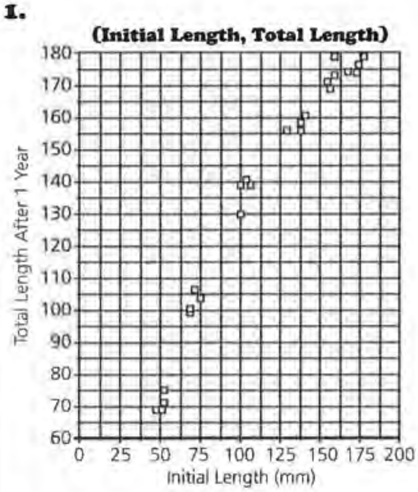
**OBJECTIVES**

Find and interpret slope as a rate of change.

Find the rate of change from data.

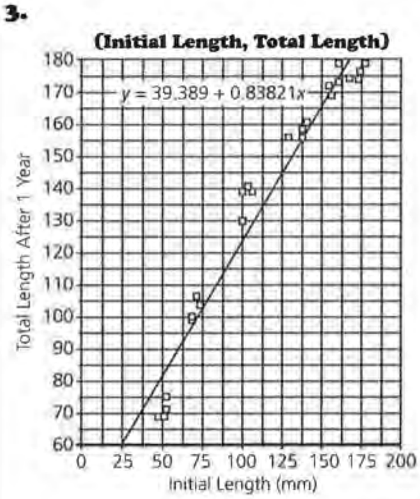
## Solution Key

**1.**

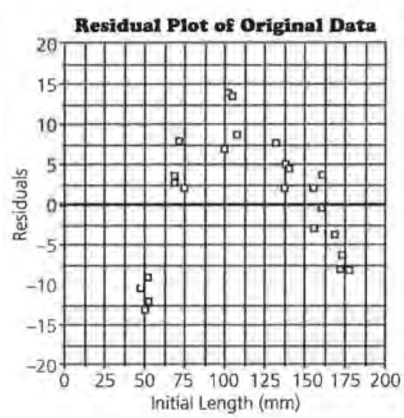



**2.** Analyzing the data from both the numeric and graphic information, it can be determined that the data are nonlinear and increasing. It also appears that the fish may be reaching some sort of limited growth, but more data would be needed to determine whether that is true.

**3.**

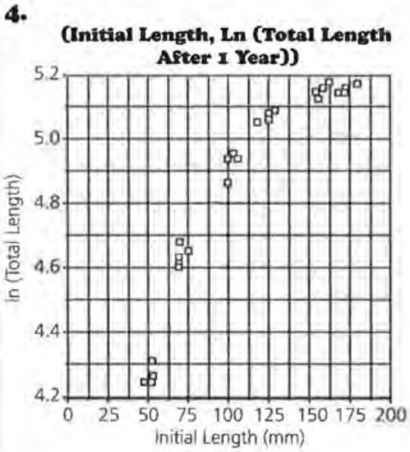



After drawing the regression line, it appears that the original plot could be considered a good fit with a correlation coefficient $r$ of 0.98 and $r^2 = 0.96$.

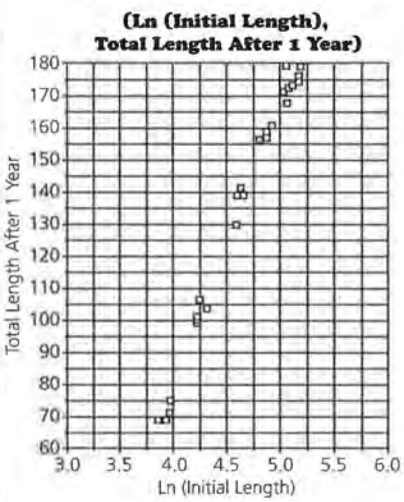



Plotting the residuals created from the regression line and the original data allows one to discover a pattern in the residuals and suggests that a transformation of the data should be performed.

**4.**

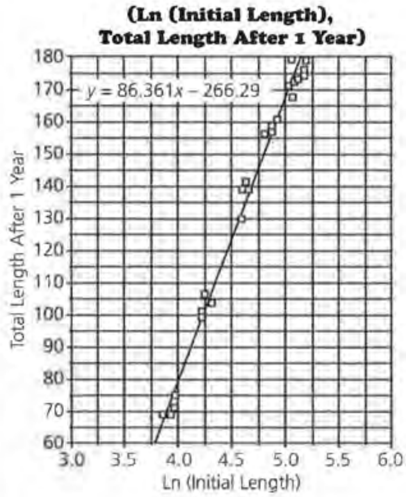



This transformation did not linearize the data, so there is no need to proceed with any analysis of it.

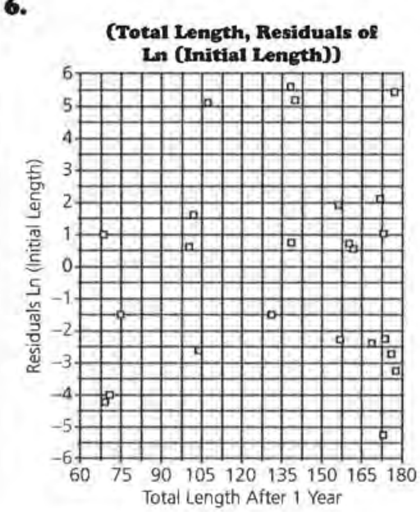



This transformation has the appearance of being linear, so proceed with a further analysis.

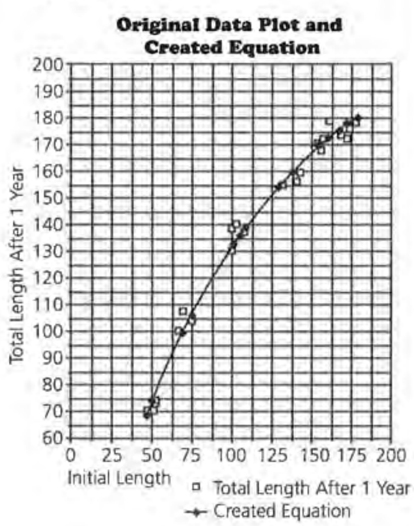
**5.** Drawing the regression line on the plot reinforces the feeling that the transformation causes the plot to become linear. The equation $L = 86.361 \ln(I) - 266.29$, where $L$ is the total length and $I$ is the initial length, has a correlation coefficient of 0.996 and $r^2 = 0.993$, suggesting a good fit.



**6.**




**7.** The residual plot confirms that the transformation does indeed linearize the data and the equation. $L = 86.361\ \ln(I) - 266.29$ should be the original equation for the data.

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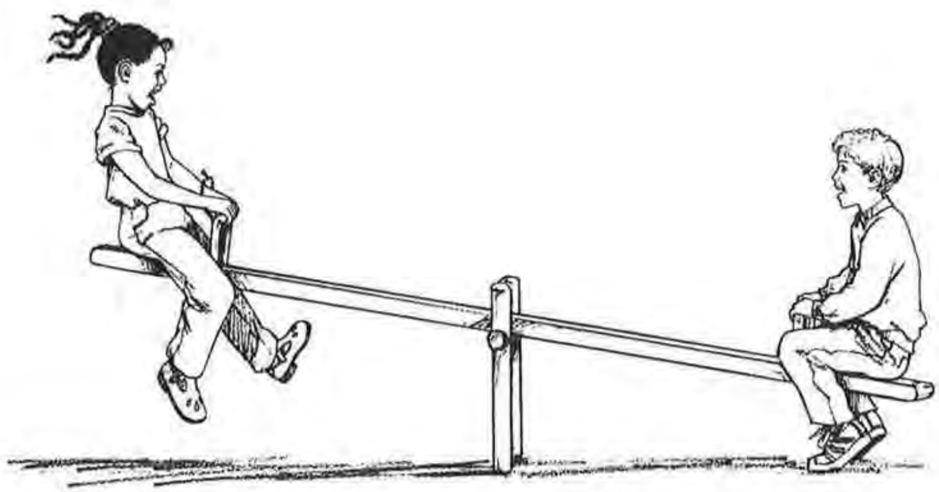
6. Plot the residuals against $x$ and consider the scale to determine a best line.
7. Use inverses of the transformations used for the equation you chose in Question 5 to create an equation that will be the best predictor equation of the original data set.

# Teacher Resources

UNIT I TEST

## Patterns and Scale Changes

NAME ______________________________



**1.** A seesaw with two people on it balances when the products of each mass and corresponding distance from the fulcrum are equal:

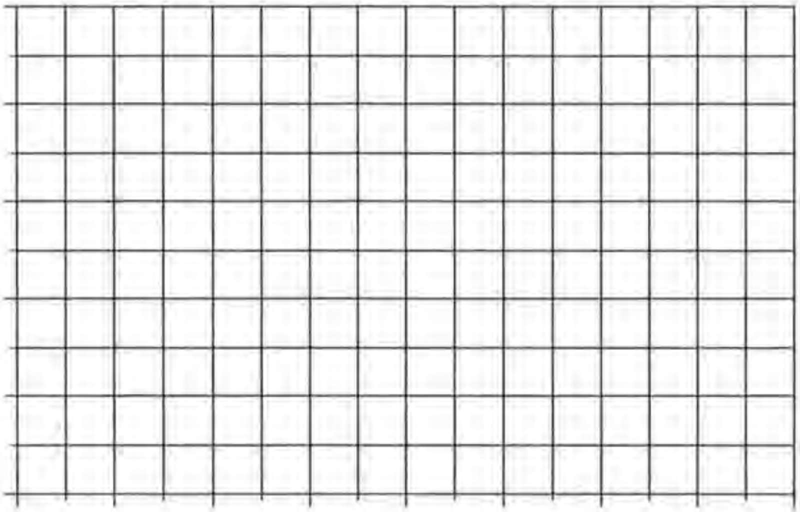
$m_1 \cdot d_1 = m_2 \cdot d_2$

**a.** Suppose $m_1 = 30$ kg and $d_1 = 1.5$ m. Generate at least eight values for $m_2$ and $d_2$ that satisfy the equation $m_1 \cdot d_1 = m_2 \cdot d_2$. Complete the following table.

| $m_2$ | $d_2$ |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |

**b.** Use words to describe the pattern.

**c.** On the grid below, graph the scatter plot of the ordered pairs $(m_2, d_2)$ you created in part a.



**d.** Through your scatter plot above, draw a straight line that appears to come closest to all of the points.

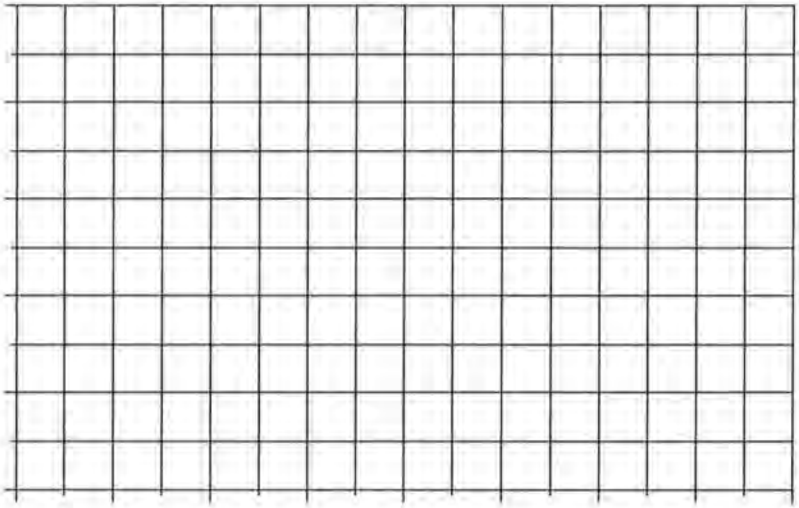
**e.** Explain why or why not a linear equation would be a good summary for the pattern.

**2.** Air pressure varies with altitude and depth. Scuba divers often dive to great depths. The following data set describes the relationship between depth in feet and underwater pressure. One atmosphere (1 atm) is the standard pressure of the air at sea level.

| Depth (ft) | Pressure (atm) |
|---|---|
| 0 | 1.0 |
| 10 | 1.3 |
| 31 | 2.0 |
| 50 | 2.6 |
| 100 | 4.2 |
| 282 | 10.0 |
| 437 | 15.0 |

Source: *Challenge of the Unknown Teaching Guide*

**a.** Use the data to graph the scatter plot of (depth, pressure) on the grid below.



**b.** Through your scatter plot in part a, draw a straight line that appears to come closest to all of the points.

**c.** Explain why or why not a linear equation would be a good summary for the pattern.

**d.** Find the equation of the line you drew for this data.

**e.** What does the slope of this line tell you about the pattern? Explain.

**3.** The following data show the orbit time and average distance from the sun for each planet.

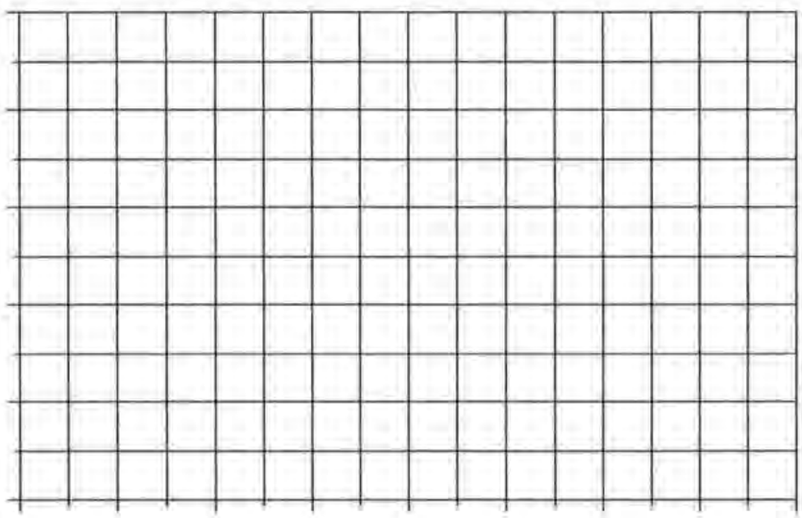
| Planet | Distance from Sun (million miles) | Orbit Time (earth years) |
|---|---|---|
| Mercury | 36 | 0.241 |
| Venus | 67.25 | 0.616 |
| Earth | 93 | 1.0 |
| Mars | 141.75 | 1.882 |
| Jupiter | 483.80 | 11.869 |
| Saturn | 887.95 | 29.660 |
| Uranus | 1764.50 | 84.044 |
| Neptune | 2791.05 | 164.140 |
| Pluto | 3653.90 | 248.833 |

**a.** Find the mean of the distances from the sun.

**b.** Find the median of the distances from the sun.

**c.** Which of the two, mean or median, would better describe the center of the distances? Why?

**d.** Use the data to graph the scatter plot of (distance from the sun, orbit time) on the grid below.



**e.** Explain why or why not a linear equation would be a good summary for the pattern.

**f.** Through your scatter plot in part d, draw a straight line that appears to come closest to all of the points.

**g.** Use mathematical symbols to describe the relationship between the orbit time measured in earth years and orbit time measured in earth days.

**h.** Describe how a scatter plot of (distance from the sun in million miles, orbit in earth days) would look different from the scatter plot you drew.

**i.** Describe how it would look the same.

## UNIT II TEST

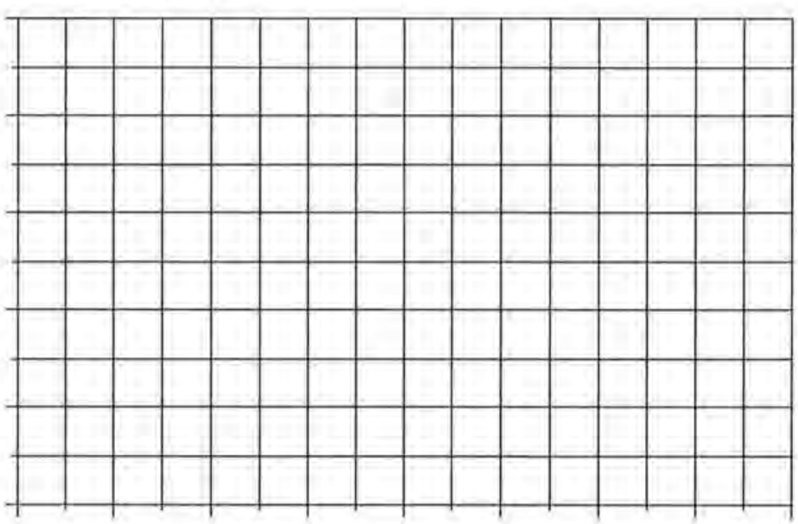
# Functions and Transformations

NAME ______________________________

**1.** Use these data collected in a high-school physics class to answer the questions.

| Time (seconds) | Height of Bounce for a Ball Dropped from 2.09 m (meters) |
|---|---|
| 0 | 2.09 |
| 1.2 | 1.52 |
| 2.2 | 1.2 |
| 3 | 0.94 |
| 3.6 | 0.76 |
| 4.2 | 0.63 |
| 4.7 | 0.57 |
| 5.2 | 0.52 |
| 5.5 | 0.49 |
| 5.9 | 0.45 |
| 6.2 | 0.44 |
| 6.5 | 0.42 |

**a.** On the grid below, draw a scatter plot of (time, height of bounce) using the data from the table.



**b.** Describe the graph family you think your graph belongs to. Justify your answer.

**c.** Draw a scatter plot of (time squared, height of bounce) on the grid below.

**d.** Draw a scatter plot of (square root of time, height of bounce) on the following grid.

**e.** Identify which of the two transformations appears to straighten the curve.

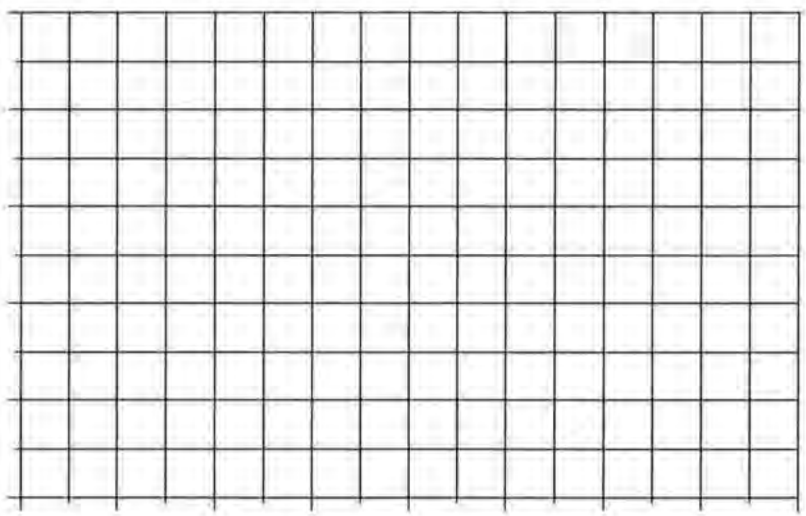
**f.** Describe your observations regarding patterns in the data and graphs.

2. The number of cellular-phone subscribers is increasing steadily. The data provided are from the Cellular Telecommunications Industry Association, Washington, D.C.

| **Year** | **Number of Subscribers (thousands)** |
|---|---|
| 1986 | 682 |
| 1987 | 1,231 |
| 1988 | 2,067 |
| 1989 | 3,509 |
| 1990 | 5,283 |
| 1991 | 7,557 |
| 1992 | 11,033 |
| 1993 | 16,009 |

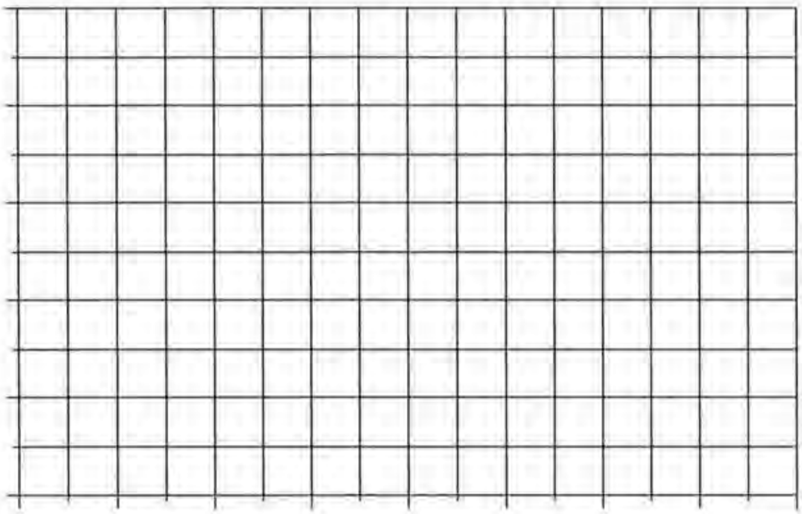
Source: *The American Almanac,* 1995

a. Draw a scatter plot of (year, number of subscribers) on the following grid.



b. Find the equation of the line of best fit and draw it on your scatter plot in part a.

**c.** Identify a transformation that would straighten the curve. Create the table and plot the graph on the following grid.



**d.** Find the equation of the line of best fit for the transformed data and draw it on the grid above.

**e.** Use the equation from part d to predict the number of cellular-phone subscribers for the year 2005.

# Mathematical Models from Data

NAME ____________________

In 1996, NBC paid 456 million dollars for the U.S. television rights for the Olympic Games in Atlanta. While this seems like a great deal of money, the projected figure was actually 600 million dollars.

The following table show the amount paid each Olympic year for the U.S. television rights. Use the table and the steps listed below to create a mathematical model. Record information for each step.

| Year | Olympic City | Network | Cost of TV Rights (million dollars) |
|---|---|---|---|
| 1960 | Rome | CBS | 0.394 |
| 1964 | Tokyo | NBC | 1.5 |
| 1968 | Mexico City | ABC | 4.5 |
| 1972 | Munich | ABC | 13.5 |
| 1976 | Montreal | ABC | 25 |
| 1980 | Moscow | NBC | 87 |
| 1984 | Los Angeles | ABC | 225 |
| 1988 | Seoul | NBC | 300 |
| 1992 | Barcelona | NBC | 401 |
| 1996 | Atlanta | NBC | 456 |

**1.** On the grid below, draw a scatter plot of (year, cost of TV rights) from the data in the table.

2. Describe patterns in the data.

3. Apply a transformation to one of the variables in the data set.

   a. Create a table and plot the transformed data on the following grid.

   b. Write the equation of the regression line and draw it on the grid above. Record $r$ and $r^2$.

   c. Create a residual plot for the transformed data on the following grid.

   d. Record your observations regarding patterns in this residual plot.

4. Repeat Question 3 using one additional transformation.

a. Create a table and plot the transformed data on the following grid.

b. Write the equation of the regression line and draw it on the grid above. Record $r$ and $r^2$.

c. Create a residual plot for the transformed data on the following grid.

d. Record your observations regarding patterns in this residual plot.

5. Use $r$, $r^2$, and information about the residuals to choose a mathematical model.

a. Generate a model (equation) for the original data set.

b. Prepare an argument defending your model.

c. Use your model to predict the cost of U.S. television rights for the years 2000 and 2004.

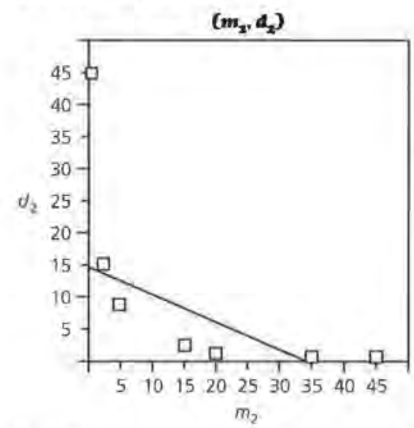
# Patterns and Scale Changes

**1. a.** Answers will vary; sample:

| $m_2$ | $d_2$ |
|---|---|
| 1 | 45 |
| 3 | 15 |
| 5 | 9 |
| 9 | 5 |
| 15 | 3 |
| 20 | 2.25 |
| 35 | 1.29 |
| 45 | 1 |

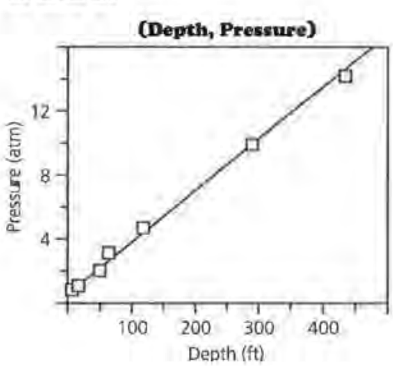
**b.** As the mass increases, the distance decreases. The product of the mass and distance is 45 units. This is an inverse variation.

**c. and d.** Graphs will vary. Sample based on answers in part a:




**e.** The plot appears to be curved. The influence of the two endpoints is not captured by a straight line.

**2. a. and b.**




**c.** The data all lie close to the line; therefore, it would be a good summary.

**d.** $y = 0.032x + 0.995$

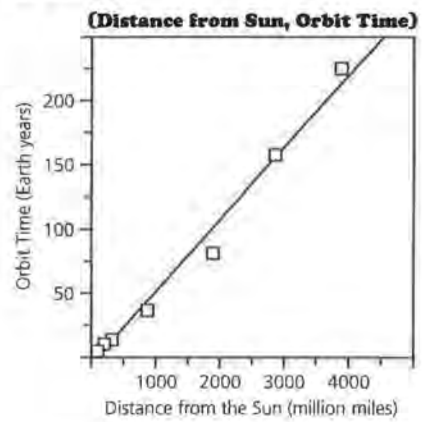
**e.** For every increase of depth by 1 foot, the pressure increases 0.032 atmospheres.

**3. a.** 1102.13 million miles

**b.** 483.8 million miles

**c.** The median would better describe the center, because the distances of Pluto and Neptune, as outliers, increase the value of the mean a great deal.

**d. and f.**




**e.** It appears that the data are in a curved rather than a linear pattern. However, the distance the points appear to be from the line isn't too great and a line could be used as a summary with the knowledge that it is not exact.

**f.** See graph above; possible equation: $y = 0.066x - 12.596$

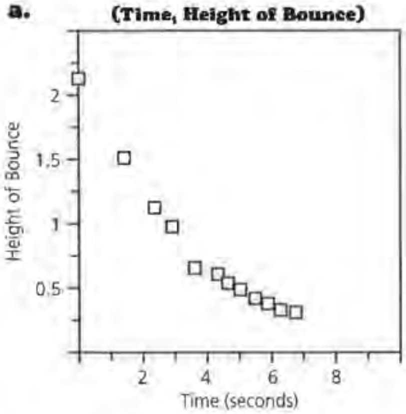
**g.** $d = 365y$

**h.** The graph would be stretched vertically.

**i.** The shape of the graph would remain the same.

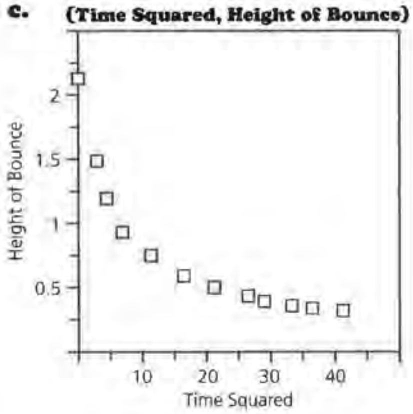
# Functions and Transformations

**1. a.** **(Time, Height of Bounce)**

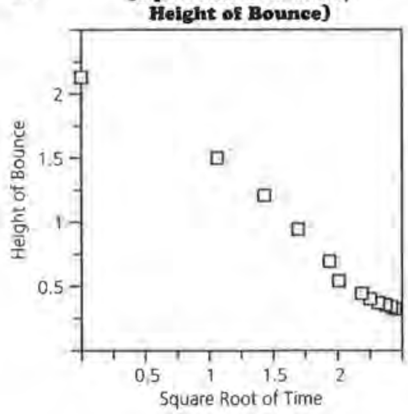



**b.** The graph appears to be a decreasing quadratic function.

**c.** **(Time Squared, Height of Bounce)**




**d.** **(Square Root of Time, Height of Bounce)**

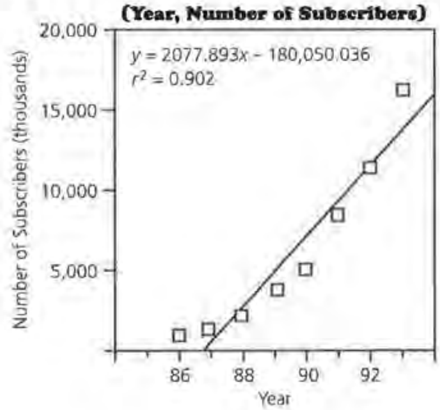



**e.** The square-root-of-time transformation had a straightening effect.

**f.** This appears to be a decreasing function. As time increases, the distance decreases. Since the distance is affected by gravity, which is a quadratic relation, the square-root transformation should straighten the graph.

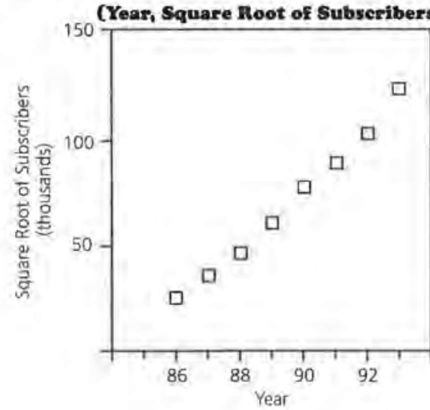
**2. a. and b.**

**(Year, Number of Subscribers)**

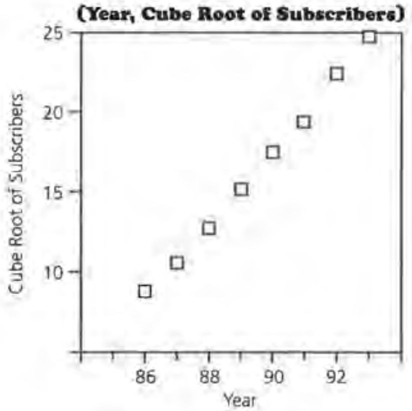



**c.** Answers will vary; two samples are given here.

**(Year, Square Root of Subscribers)**



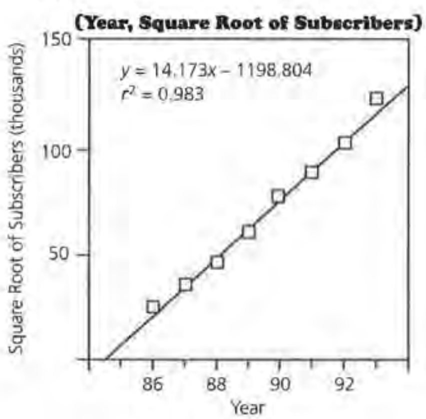

| Year | $\sqrt{\text{Subscribers}}$ |
|---|---|
| 86 | 26.1 |
| 87 | 35.1 |
| 88 | 45.5 |
| 89 | 59.2 |
| 90 | 72.7 |
| 91 | 86.9 |
| 92 | 105.0 |
| 93 | 126.5 |



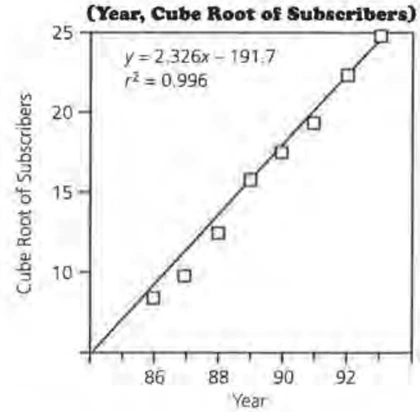

| Year | $\sqrt[3]{\text{Subscribers}}$ |
|---|---|
| 86 | 8.8 |
| 87 | 10.7 |
| 88 | 12.7 |
| 89 | 15.2 |
| 90 | 17.4 |
| 91 | 19.6 |
| 92 | 22.3 |
| 93 | 25.2 |

**d.** Answers will depend upon the transformation chosen in part c.

For the square-root transformation, $y = 14.173x - 1198.8$, letting $x = 86$ for the year 1986.



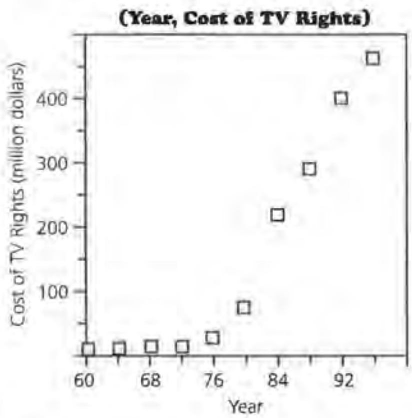

For the cube root transformation, $y = 2.326x - 191.7$, letting $x = 86$ for the year 1986.




**e.** Answers will depend upon the transformation chosen in part c. The cube root transformation predicts about 145 million subscribers in the year 2005.

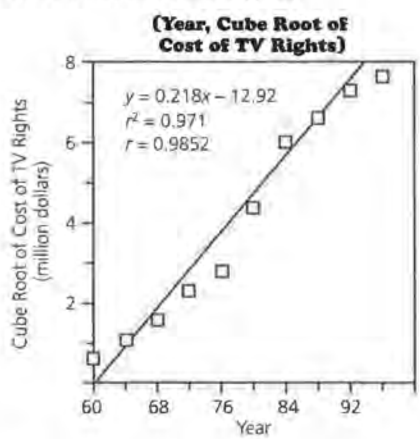
# Mathematical Models from Data

**1.**



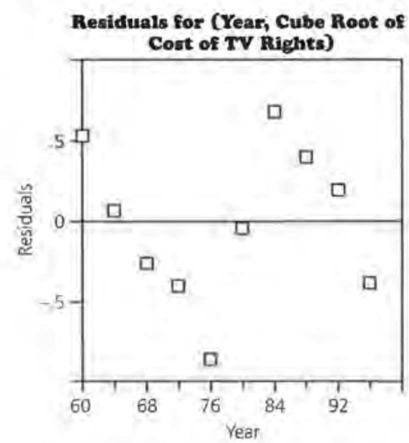

**2.** The data are increasing at a rate that is not constant.

**3.** **a. and b.** Answers will vary. Sample: (year, cube root of cost of TV rights)




| Year | Cube Root of TV Costs |
|---|---|
| 60 | 0.733 |
| 64 | 1.145 |
| 68 | 1.651 |
| 72 | 2.381 |
| 76 | 2.924 |
| 80 | 4.431 |
| 84 | 6.082 |
| 88 | 6.694 |
| 92 | 7.374 |
| 96 | 7.697 |

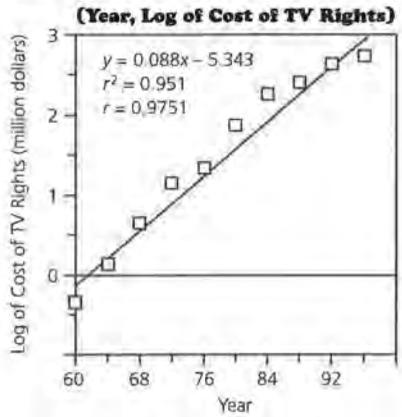
**c.** Answers will vary. For the transformation chosen in part a, the residual plot is:




| $x$ | Actual $y$ | Predicted $y$ | $r$ |
|---|---|---|---|
| 60 | 0.733 | 0.16 | 0.573 |
| 64 | 1.145 | 1.032 | 0.113 |
| 68 | 1.651 | 1.904 | −0.253 |
| 72 | 2.381 | 2.776 | −0.395 |
| 76 | 2.924 | 3.648 | −0.724 |
| 80 | 4.431 | 4.52 | −0.089 |
| 84 | 6.082 | 5.392 | 0.69 |
| 88 | 6.694 | 6.264 | 0.43 |
| 92 | 7.374 | 7.136 | 0.238 |
| 96 | 7.697 | 8.008 | −0.311 |

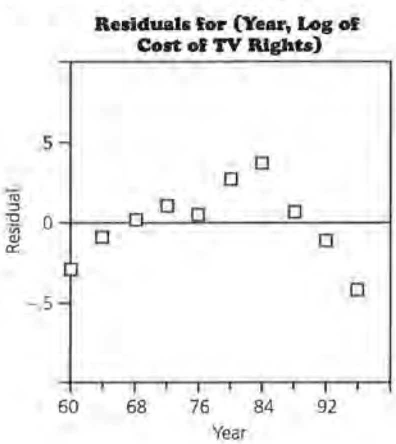
**d.** Answers will depend on the transformation chosen in part a. The residuals for the transformation shown here do not seem to fit any pattern.

**4. a. and b.** Answers will vary. Sample: (year, log of cost of TV rights)




| Year | Log of TV Cost |
|---|---|
| 60 | –0.405 |
| 64 | 0.176 |
| 68 | 0.653 |
| 72 | 1.130 |
| 76 | 1.398 |
| 80 | 1.940 |
| 84 | 2.352 |
| 88 | 2.477 |
| 92 | 2.603 |
| 96 | 2.659 |

**c.** Answers will vary. For the transformation chosen above, the residual plot is:




| $x$ | Actual $y$ | Predicted $y$ | $r$ |
|---|---|---|---|
| 60 | –0.405 | –0.063 | –0.342 |
| 64 | 0.176 | 0.289 | –0.113 |
| 68 | 0.653 | 0.641 | 0.012 |
| 72 | 1.130 | 0.993 | 0.137 |
| 76 | 1.398 | 1.345 | 0.053 |
| 80 | 1.940 | 1.697 | 0.243 |
| 84 | 2.352 | 2.049 | 0.303 |
| 88 | 2.477 | 2.401 | 0.076 |
| 92 | 2.603 | 2.753 | –0.150 |
| 96 | 2.659 | 3.105 | –0.446 |

**d.** Answers will depend upon the transformation chosen in part a. The residuals for the transformation shown here do not seem to fit any pattern and are relatively small in size as indicated by the scale on the vertical axis.

**5. a.** Answers will vary. The equation for the cube root model is

$y = 0.0104x^3 - 1.847x^2 + 109.32x - 2156.689.$

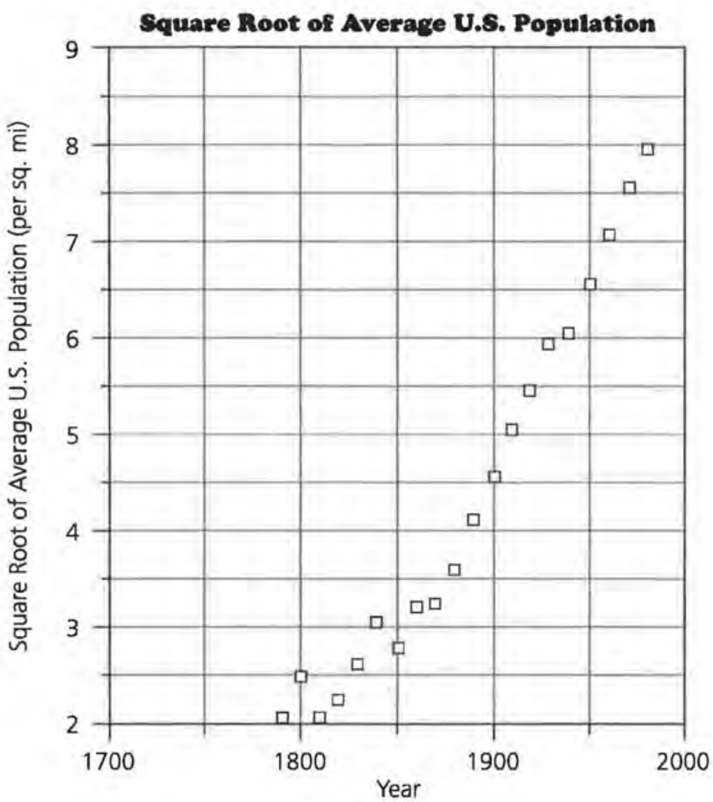
**b.** Answers will vary.

**c.** Answers will vary. Using the cube-root model, the cost will be about \$705 million in 2000 and \$934 million in 2004.

ACTIVITY SHEET 1

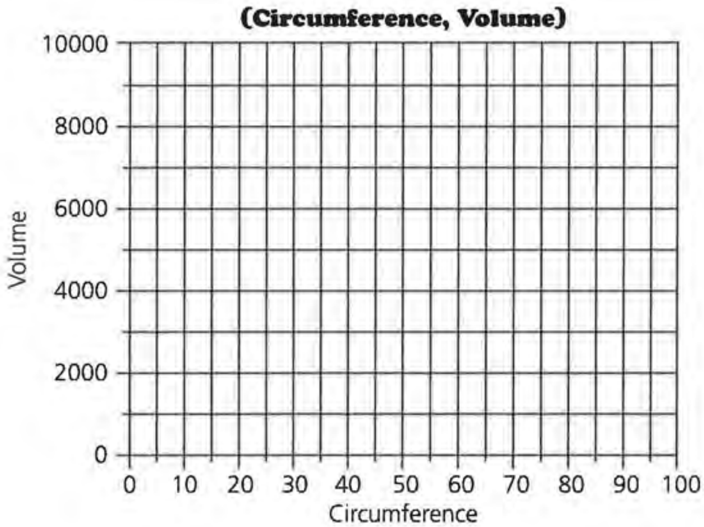
# Lesson 5, Questions 8 and 9

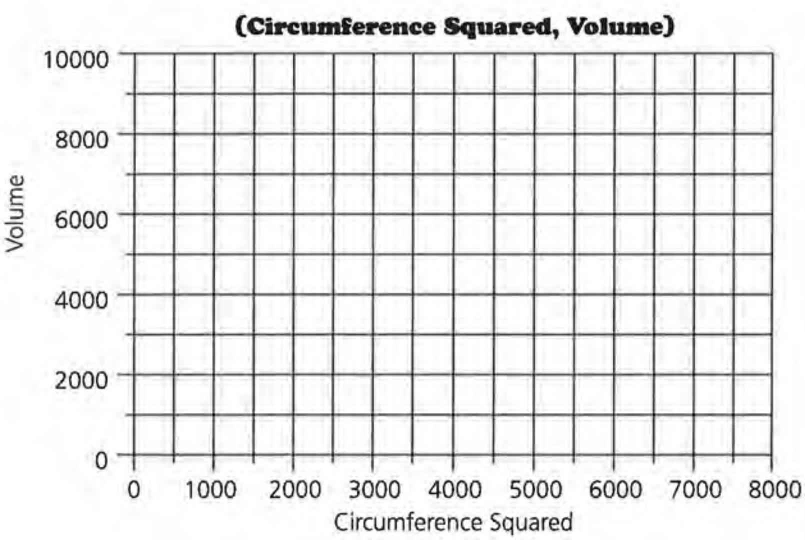
NAME ______________________________



# Lesson 6, Questions 1–8

NAME ______________________________

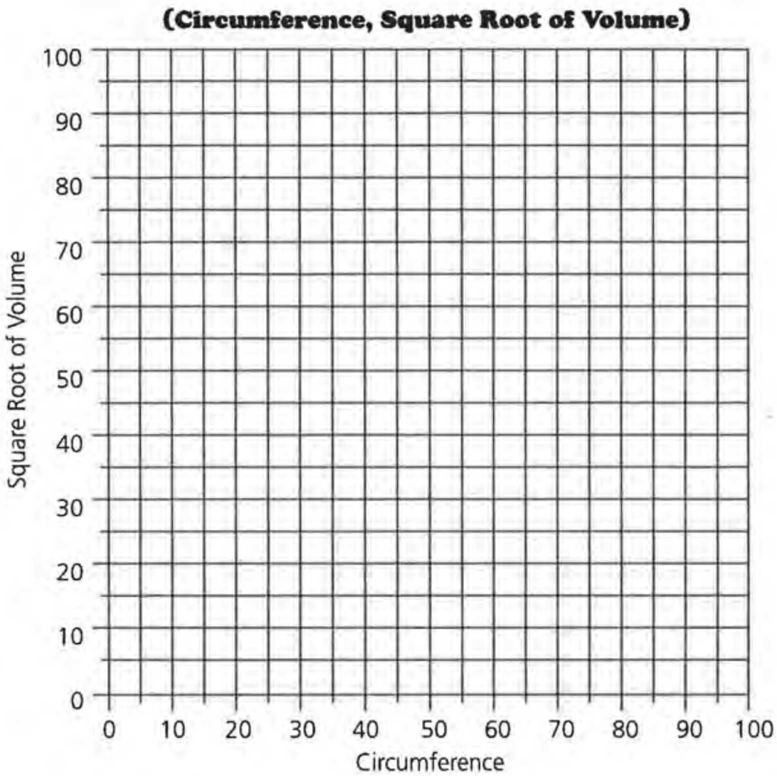


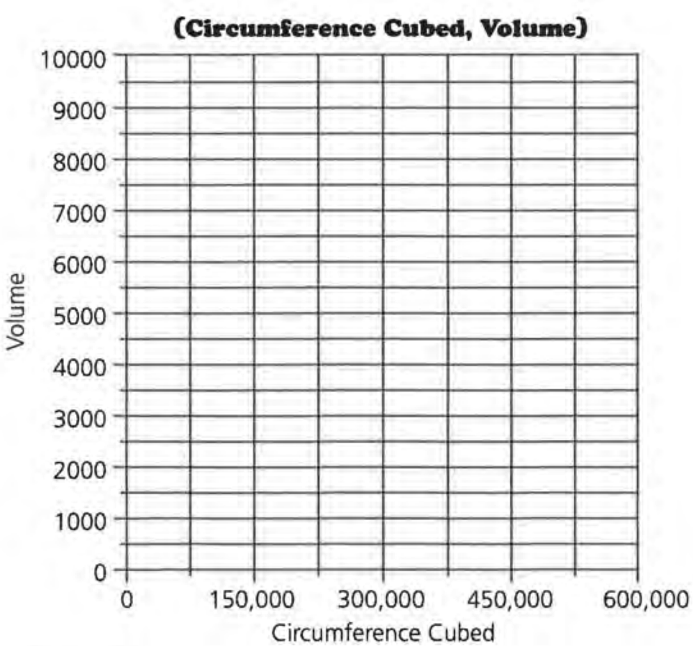



ACTIVITY SHEET 3

# Lesson 6, Questions 1–8

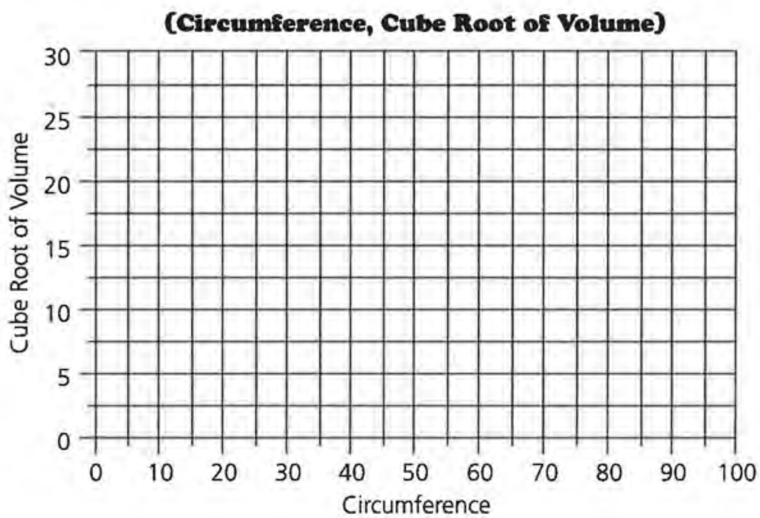
NAME ____________________



# Lesson 6, Questions 1–8

NAME ______________________________



**Data-Driven Mathematics**
**Procedures for Using the TI-83**

## I. Clear menus

**ENTER** will execute any command or selection. Before beginning a new problem, previous instructions or data should be cleared. Press **ENTER** after each step below.

1. To clear the function menu, **Y=**, place the cursor anyplace in each expression, **CLEAR**
2. To clear the list menu, **2nd MEM ClrAllLists**
3. To clear the draw menu, **2nd DRAW ClrDraw**
4. To turn off any statistics plots, **2nd STATPLOT PlotsOff**
5. To remove user created lists from the Editor, **STAT SetUpEditor**

## II. Basic information

1. A rule is active if there is a dark rectangle over the option. See Figure 1.

```
Plot1 Plot2 Plot3
\Y1 = 170 + 7(X–72)
\Y2 = –.35X – 22
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

FIGURE 1

On the screen above, Y1 and Plot1 are active; Y2 is not. You may toggle Y1 or Y2 from active to inactive by putting the cursor over the = and pressing **ENTER**. Arrow up to **Plot1** and press **ENTER** to turn it off; arrow right to **Plot2** and press **ENTER** to turn it on, etc.

2. The Home Screen (Figure 2) is available when the blinking cursor is on the left as in the diagram below. There may be other writing on the screen. To get to the Home Screen, press **2nd QUIT**. You may also clear the screen completely by pressing **CLEAR**.

```
7→X
                     7
Y1
                    11
■
```

FIGURE 2

3. The variable $x$ is accessed by the **X, T, Θ, *n*** key.
4. Replay option: **2nd ENTER** allows you to back up to an earlier command. Repeated use of **2nd ENTER** continues to replay earlier commands.
5. Under **MATH**, the **MATH** menu has options for fractions to decimals and decimals to fractions, for taking *n*th roots, and for other mathematical operations. **NUM** contains the absolute value function as well as other numerical operations. (Figure 3)

```
MATH NUM CPX PRB
1: abs(
2: round(
3: iPart(
4: fPart(
5: int(
6: min(
7↓max(
```

FIGURE 3

## III. The STAT Menus

1. There are three basic menus under the **STAT** key: **EDIT, CALC,** and **TESTS.** Data are entered and modified in the **EDIT** mode; all numerical calculations are made in the **CALC** mode; statistical tests are run in the **TEST** mode.
2. **Lists and Data Entry**
   Data is entered and stored in Lists (Figure 4). Data will remain in a list until the list is cleared. Data can be cleared using **CLEAR** $L_i$ or (List name), or by placing the cursor over the List heading and selecting **CLEAR ENTER.** To enter data, select **STAT EDIT** and with the arrow keys move the cursor to the list you want to use.

Type in a numerical value and press **ENTER.** Note that the bottom of the screen indicates the List you are in and the list element you have highlighted. 275 is the first entry in L1. (It is sometimes easier to enter a complete list before beginning another.)

| L1 | L2 | L3 |
|---|---|---|
| 275 | 67 | 190 |
| 5311 | 144 | 120 |
| 114 | 64 | 238 |
| 2838 | 111 | 153 |
| 15 | 90 | 179 |
| 332 | 68 | 207 |
| 3828 | 94 | 153 |
| L1 (1) = 275 | | |

FIGURE 4

For data with varying frequencies, one list can be used for the data, and a second for the frequency of the data. In Figure 5 below, the L5(7) can be used to indicate that the seventh element in list 5 is 4, and that 25 is a value that occurs 4 times.

| L4 | L5 | L6 |
|---|---|---|
| 55 | 1 | -------- |
| 50 | 3 | |
| 45 | 6 | |
| 40 | 14 | |
| 35 | 12 | |
| 30 | 9 | |
| 25 | 4 | |
| L5 (7) = 4 | | |

FIGURE 5

3. **Naming Lists**
Six lists are supplied to begin with. L1, L2, L3, L4, L5, and L6 can be accessed also as **2nd $L_i$**. Other lists can be named using words as follows. Put the cursor at the top of one of the lists. Press **2nd INS** and the screen will look like that in Figure 6.

| | L1 | L2 1 |
|---|---|---|
| | -------- | -------- |
| Name = | | |

FIGURE 6

The alpha key is on, so type in the name (up to five characters) and press **ENTER.** (Figure 7)

| PRICE | L1 | L2 2 |
|---|---|---|
| | -------- | -------- |
| PRICE(1) = | | |

FIGURE 7

Then enter the data as before. (If you do not press **ENTER,** the cursor will remain at the top and the screen will say "error: data type.") The newly named list and the data will remain until you go to Memory and delete the list from the memory. To access the list for later use, press **2nd LIST** and use the arrow key to locate the list you want under the **NAMES** menu. You can accelerate the process by typing **ALPHA P** (for price). (Figure 8) Remember, to delete all but the standard set of lists from the editor, use SetUp Editor from the **STAT** menu.

NAMES OPS MATH
↑PRICE
: RATIO
: RECT
: RED
: RESID
: SATM
↓SATV

FIGURE 8

## 4. Graphing Statistical Data

**General Comments**

- Any graphing uses the **GRAPH** key.
- Any function entered in Y1 will be graphed if it is active. The graph will be visible only if a suitable viewing window is selected.
- The appropriate *x*- and *y*-scales can be selected in **WINDOW**. Be sure to select a scale that is suitable to the range of the variables.

**Statistical Graphs**

To make a statistical plot, select **2nd Y=** for the **STAT PLOT** option. It is possible to make three plots concurrently if the viewing windows are identical. In Figure 9, Plots 2 and 3 are off, Plot 1 is a scatter plot of the data (Costs, Seats), Plot 2 is a scatter plot of (L3,L4), and Plot 3 is a box plot of the data in L3.

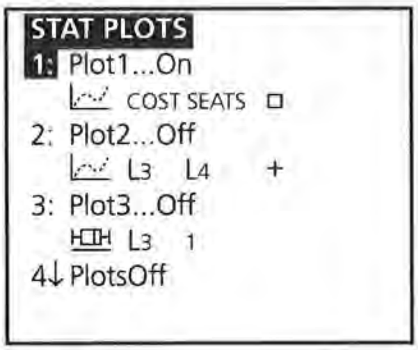



FIGURE 9

Activate one of the plots by selecting that plot and pressing **ENTER.**

Choose **ON,** then use the arrow keys to select the type of plot (scatter, line, histogram, box plot with outliers, box plot, or normal probability plot). (In a line plot, the points are connected by segments in the order in which they are entered. It is best used with data over time.) Choose the lists you wish to use for the plot. In the window below, a scatter plot has been selected with the *x*-coordinate data from COSTS, and the *y*-coordinate data from SEATS. (Figure 10) (When pasting in list names, press **2nd LIST,** press **ENTER** to activate the name, and press **ENTER** again to locate the name in that position.)

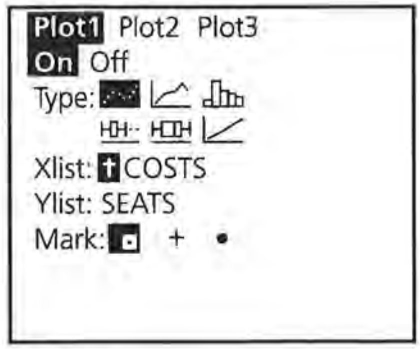



FIGURE 10

For a histogram or box plot, you will need to select the list containing the data and indicate whether you used another list for the frequency or are using 1 for the frequency of each value. The *x*-scale selected under **WINDOW** determines the width of the bars in the histogram. It is important to specify a scale that makes sense with the data being plotted.

## 5. Statistical Calculations

One-variable calculations such as mean, median, maximum value of the list, standard deviation, and quartiles can be found by selecting **STAT CALC 1-Var Stats** followed by the list in which you are interested. Use the arrow to continue reading the statistics. (Figures 11, 12, 13)

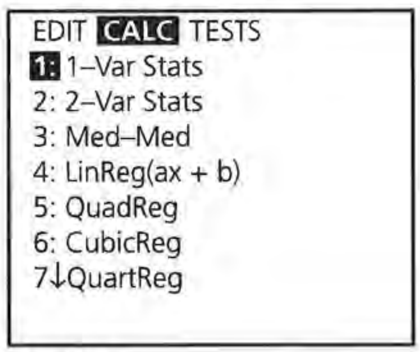



FIGURE 11

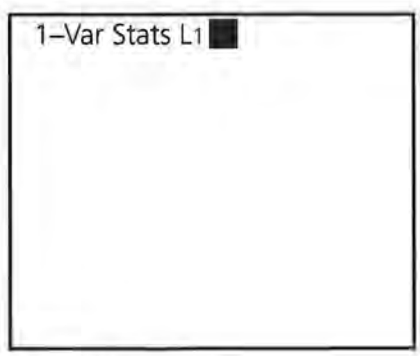



FIGURE 12

```
1–Var Stats
x̄ = 1556.20833
Σx = 37349
Σx² = 135261515
Sx = 1831.353621
σx = 1792.79449
↓n = 24
█
```

FIGURE 13

Calculations of numerical statistics for bivariate data can be made by selecting two variable statistics. Specific lists must be selected after choosing the **2-Var Stats** option. (Figure 14)

```
2–Var Stats L1, L
2█
```

FIGURE 14

Individual statistics for one- or two-data sets can be obtained by selecting **VARS Statistics,** but you must first have calculated either 1-Var or 2-Var Statistics. (Figure 15)

```
XY Σ EQ TEST PTS
1: n
2: x̄
3: Sx
4: σx
5: ȳ
6: Sy
7↓σy
```

FIGURE 15

6. **Fitting Lines and Drawing Their Graphs**
Calculations for fitting lines can be made by selecting the appropriate model under **STAT CALC: Med-Med** gives the median fit regression, **LinReg** the least-squares linear regression, and so on. (Note the only difference between **LinReg (ax+b)** and **LinReg (a+bx)** is the assignment of the letters a and b.) Be sure to specify the appropriate lists for $x$ and $y$. (Figure 16)

```
Med–Med ʟCal, ʟFA
CAL █
```

FIGURE 16

To graph a regression line on a scatter plot, follow the steps below:

- Enter your data into the Lists.
- Select an appropriate viewing window and set up the plot of the data as above.
- Select a regression line followed by the lists for $x$ and $y$, **VARS Y-VARS Function** (Figures 17, 18) and the $Y_i$ you want to use for the equation, followed by **ENTER.**

```
VARS Y-VARS
1: Function...
2: Parametric...
3: Polar...
4: On/Off...
```

FIGURE 17

```
Med–Med _CAL, ʟFA
CAL, Y1
```

FIGURE 18

The result will be the regression equation pasted into the function Y1. Press **GRAPH** and both the scatter plot and the regression line will appear in the viewing window. (Figures 19, 20)

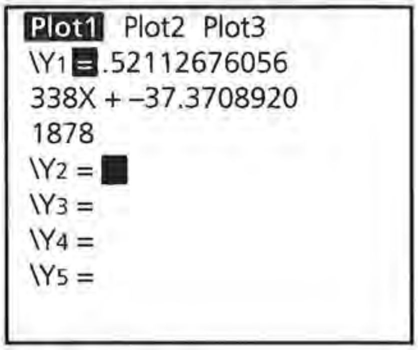



FIGURE 19

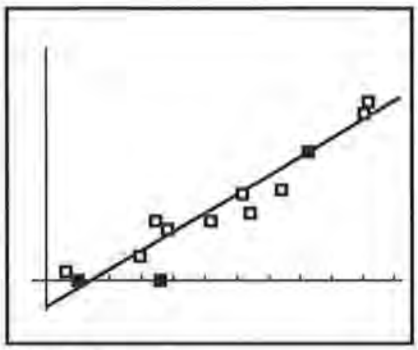


FIGURE 20

- There are two cursors that can be used in the graphing screen.

**TRACE** activates a cursor that moves along either the data (Figure 21) or the function entered in the Y-variable menu (Figure 22). The coordinates of the point located by the cursor are given at the bottom of the screen.

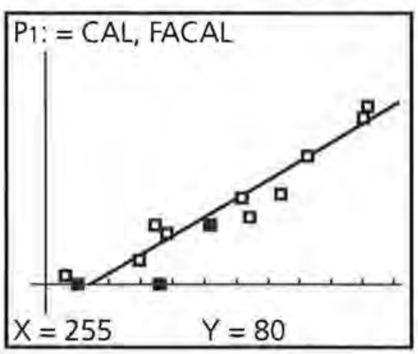



FIGURE 21

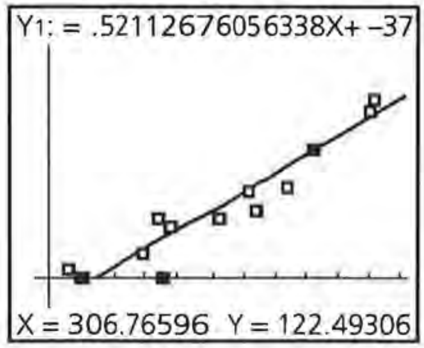



FIGURE 22

Pressing **GRAPH** returns the screen to the original plot. The up arrow key activates a cross cursor that can be moved freely about the screen using the arrow keys. See Figure 23.

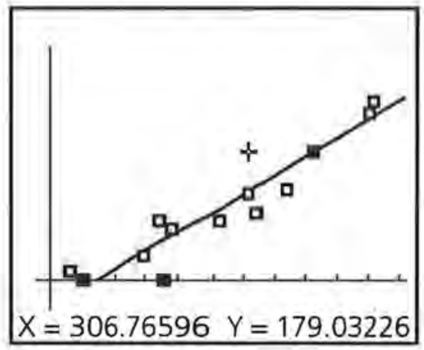



FIGURE 23

Exact values can be obtained from the plot by selecting **2nd CALC** Value. Select **2nd CALC Value ENTER.** Type in the value of *x* you would like to use, and the exact ordered pair will appear on the screen with the cursor located at that point on the line. (Figure 24)

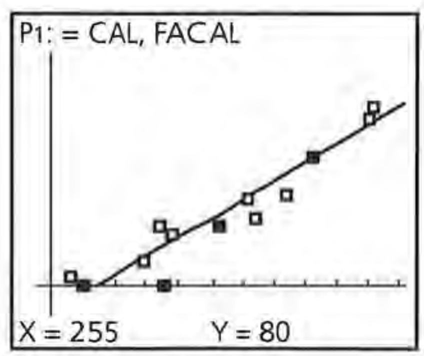



FIGURE 24

## IV. Evaluating an expression

To evaluate $y = .225\,x - 15.6$ for $x = 17$, 20, and 24, you can:

1. Type the expression in Y1, return to the home screen, 17 **STO X,T,Θ,n ENTER, VARS Y-VARS Function Y1 ENTER ENTER**. (Figure 25)

```
17 → X
                    17
Y1
           -11.775
█
```

FIGURE 25

Repeat the process for $x = 20$ and 24.

2. Type $17^2 - 4$ for $x = 17$, **ENTER** (Figure 26). Then use **2nd ENTRY** to return to the arithmetic line. Use the arrows to return to the value 17 and type over to enter 20.

```
.225*17 – 15.6
                 -11.775
.225*2█ – 15.6
```

FIGURE 26

You can also find the value of $x$ by using the table command. Select **2nd TblSet** (Figure 27). (Y1 must be turned on.) Let **TBlStart** = 17, and the increment **ΔTbl** = 1.

```
TABLE SETUP
  TblStart = 17
  ΔTbl = █
Indpnt:  Auto  Ask
Depend:  Auto  Ask
```

FIGURE 27

Select **2nd TABLE** and the values of $x$ and $y$ generated by the equation in Y1 will be displayed. (Figure 28)

| X | Y1 | |
|---|---|---|
| 17 | -11.78 | |
| 18 | -11.55 | |
| 19 | -11.33 | |
| 20 | -11.1 | |
| 21 | -10.88 | |
| 22 | -10.65 | |
| 23 | -10.43 | |
| X = 17 | | |

FIGURE 28

## V. Operating with Lists

1. A list can be treated as a function and defined by placing the cursor on the label above the list entries. List 2 can be defined as L1 + 5. (Figure 29)

| L1 | L2 | L3 |
|---|---|---|
| 275 | -------- | 190 |
| 5311 | | 120 |
| 114 | | 238 |
| 2838 | | 153 |
| 15 | | 179 |
| 332 | | 207 |
| 3828 | | 153 |
| L2 = L1 + 5 | | |

FIGURE 29

Pressing **ENTER** will fill List 2 with the values defined by L1+5. (Figure 30)

| L1 | L2 | L3 |
|---|---|---|
| 275 | 280 | 190 |
| 5311 | 5316 | 120 |
| 114 | 119 | 238 |
| 2838 | 2843 | 153 |
| 15 | 20 | 179 |
| 332 | 337 | 207 |
| 3828 | 3833 | 153 |

L2(1) = 280

FIGURE 30

2. List entries can be cleared by putting the cursor on the heading above the list, and selecting **CLEAR** and **ENTER.**

3. A list can be generated by an equation from Y= over a domain specified by the values in L1 by putting the cursor on the heading above the list entries. Select **VARS Y-VARS Function Y1 ENTER ( L1) ENTER.** (Figure 31)

| L1 | L2 | L3 |
|---|---|---|
| 120 | 12 | -------- |
| 110 | 14 | |
| 110 | 12 | |
| 110 | 11 | |
| 100 | ? | |
| 100 | 6 | |
| 120 | 9 | |

L3 = Y1(L1)

FIGURE 31

4. The rule for generating a list can be attached to the list and retrieved by using quotation marks (**ALPHA** +) around the rule. (Figure 32) Any change in the rule (Y1 in the illustration) will result in a change in the values for L1. To delete the rule, put the cursor on the heading at the top of the list, press **ENTER**, and then use the delete key. (Because L1 is defined in terms of **CAL**, if you delete **CAL** without deleting the rule for L1 you will cause an error.)

| CAL | FACAL | L1 5 |
|---|---|---|
| 255 | 80 | -------- |
| 305 | 120 | |
| 410 | 180 | |
| 510 | 250 | |
| 320 | 90 | |
| 370 | 125 | |
| 500 | 235 | |

L1 = "Y1(LCAL)"

FIGURE 32

## VI. Using the DRAW Command

To draw line segments, start from the graph of a plot, press **2ND DRAW**, and select **Line(**. (Figure 33)

DRAW POINTS STO
1: ClrDraw
2: Line(
3: Horizontal
4: Vertical
5: Tangent(
6: DrawF
7↓Shade(

FIGURE 33

This will activate a cursor that can be used to mark the beginning and ending of a line segment. Move the cursor to the beginning point and press **ENTER**; use the cursor to mark the end of the segment, and press **ENTER** again. To draw a second segment, repeat the process. (Figure 34)

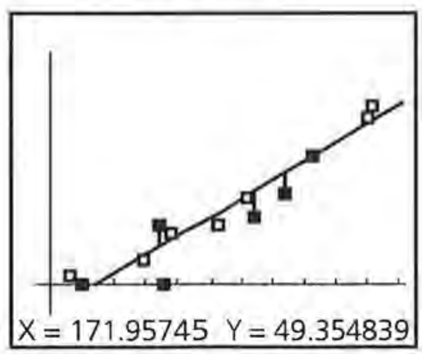



FIGURE 34

## VII. Random Numbers

To generate random numbers, press **MATH** and **PRB**. This will give you access to a random number function, **rand**, that will generate random numbers between 0 and 1 or **randInt(** that will generate random numbers from a beginning integer to an ending integer for a specified set of numbers. (Figure 35) In Figure 36, five random numbers from 1 to 6 were generated.

```
MATH NUM CPX PRB
1: rand
2: nPr
3: nCr
4: !
5: randInt(
6: randNorm(
7: randBin(
```

FIGURE 35

```
randInt(1, 6, 5,)
     (2 4 2 5 3)
```

FIGURE 36

Pressing **ENTER** will generate a second set of random numbers.

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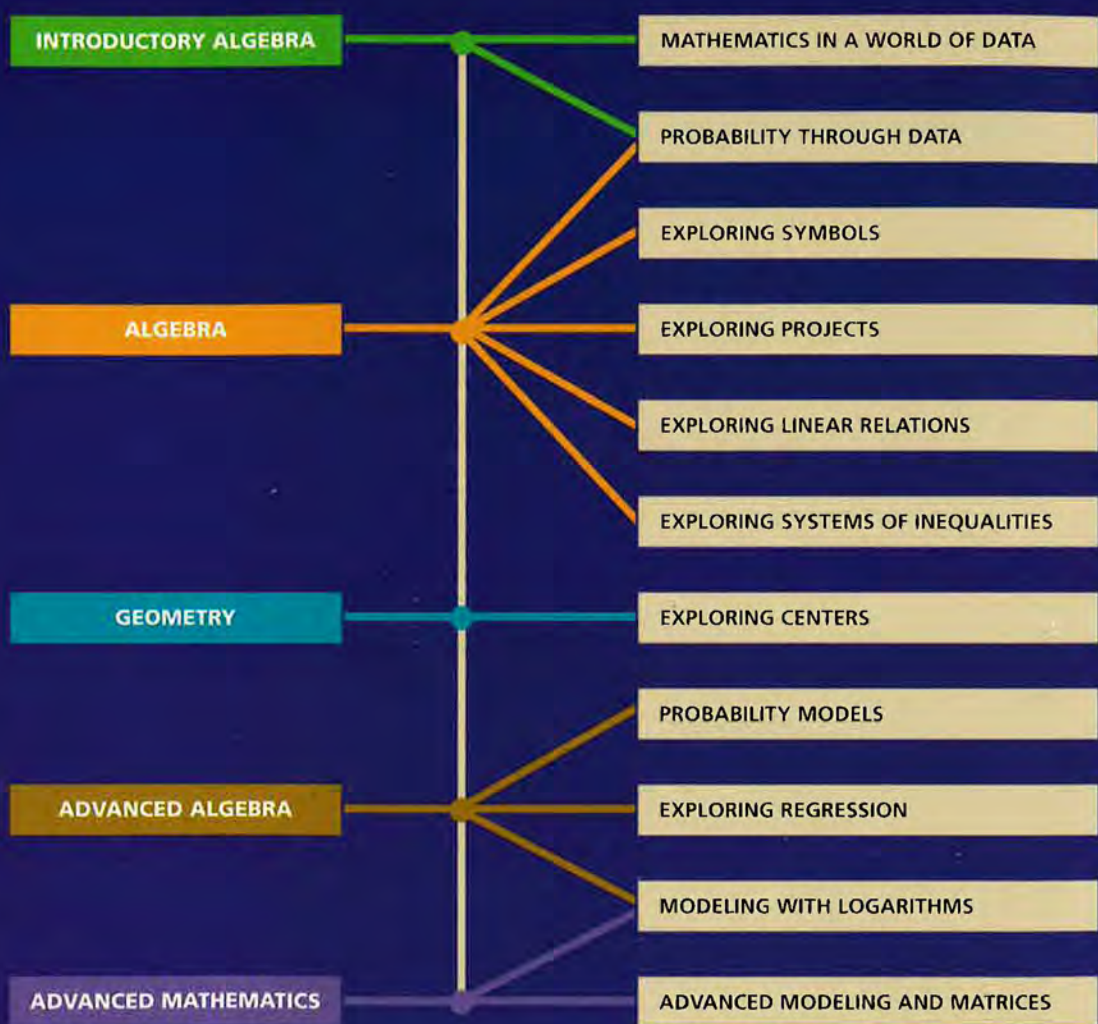


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